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ABSTRACT *

In expositions of the theory of relativity it is customary to emphasize the differences between the Newtonian and Einsteinian theories of space, time and gravitation. On close inspection it appears that these disparities are due, in part, to the differences between the languages used to express the theories. To every physical theory there corresponds a certain mathematical formalism in which the theory is usually represented. For the purpose of comparing different theories it is desirable to formulate them in the same mathematical language. Otherwise it is rather difficult to ascertain what are the relationships between the basic assumption underlying these theories.

Relativistic theories of space-time are most naturally expressed in terms of concepts from differential geometry. Following Cartan and Friedrichs, we analyze the geometrical structure of space-time in Newtonian mechanics and compare it with that in relativity. It turns out that there are a number of elements common to all theories of space-time: the basic manifold is always assumed to be a four-dimensional differentiable continuum, endowed with an affine connection. In the theory of relativity, space-time is simply a Riemannian manifold; in Newton's theory the metric structure is more complicated.

The Newtonian metric is degenerate; clearly, it is the limit, as $c \to \infty$, of the relativistic metric. Accordingly, the Newtonian metric has those properties of the relativistic g_{ab} which are preserved by the limiting process. In particular, it is invariant by parallel transport.

The Newtonian mechanics is based on the assumption that there

* The full text of this lecture is contained in a volume edited by B. Hoffmann and dedicated to Vaclav Hlavatý; to be published in 1965.

exists an absolute time, t, and that the hypersurfaces ("spaces") $t =$ const. are three-dimensional Euclidean. The time t can be taken as one of the coordinates; if (x, y, z, t) is a system of coordinates in space-time, the motion of a particle can be represented by $x = \xi(t)$, $y = \eta(t)$, $z = \zeta(t)$, i.e., by a curve (world-line) in space-time. Neglecting gravitation, the first law of dynamics may be formulated as follows: there exists a family of privileged motions, called free motions, and a system of coordinates (x, y, z, t) such that the free motions are characterized by

$$\frac{d^2x}{dt^2} = \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2} = 0$$

Coordinate systems whose existence is asserted by the first law are called inertial. A transformation leading from one inertial system to another is called Galilean. Clearly, if we agree to consider the world-lines of free motions as geodesics, the Newtonian space-time becomes endowed with an integrable affine connection.

In a gravitational field, there are no free motions in the previous sense. The best one can do is to remove all non-gravitational interactions and to consider free falls as the family of privileged motions. Accordingly, Newton's first law may be rephrased as follows: there exists a family of privileged motions, called free falls, a system of coordinates (x, y, z, t) and a function $\varphi\,(x, y, z, t)$ such that the free falls are characterized by

$$\frac{d^2x}{dt^2} = -\frac{\partial\varphi}{\partial x}; \qquad \frac{d^2y}{dt^2} = -\frac{\partial\varphi}{\partial y}; \qquad \frac{d^2z}{dt^2} = -\frac{\partial\varphi}{\partial z}$$

Clearly, the class of coordinate changes preserving these equations is much larger than the class of Galilean transformations. Usually, one considers gravitational fields produced by bounded sources. One then can normalize φ by requiring that it vanishes at large distances; this eliminates the possibility of more general transformations and restores the privileged role of the Galilei group. However, this cannot be done when there is a strong gravitational field extending all over space, as in cosmology. In any case, we may call world-lines corresponding to free falls geodesics, and thereby introduce an affine connection in space-time.

The general geometrical structure of space-time in relativity is very well known. In the Newtonian theory the space-time is a differentiable manifold N of class C_∞, homeomorphic to R^4. The Newtonian notion of absolute simultaneity implies the existence of a family T of hypersurfaces in space-time. Distinct elements of T do not intersect, through every event (point of N) there passes an element of T; all these hypersurfaces have the topology of R^3. Let $t =$ const. be the equation of T.

The family of all free falls determines a symmetric affine connection Γ^a_{bc} on N. It follows from the first law of dynamics that t can be chosen so as to be an affine parameter along all time-like geodesics. This defines t up to linear transformations; every such t is called the absolute time.

Let (y^1, y^2, y^3) be a system of local coordinates in a hypersurface belonging to T. The hypersurface can be represented in a parametric form

$$x^a = x^a(y^1, y^2, y^3)$$

Let $h^{\alpha\beta}$ denote the components with respect to $(\partial/\partial y^\alpha)$ of the Euclidean metric tensor of the hypersurface. Its components with respect to $(\partial/\partial x^a)$ are

$$h^{ab} = h^{\alpha\beta}\frac{\partial x^a}{\partial y^\alpha}\frac{\partial x^b}{\partial y^\beta}$$

Clearly

$$h^{ab}t_b = 0 \text{ where } t_b = \partial_b t$$

and the canonical form of the matrix (h^{ab}) is diag (1, 1, 1, 0). The tensor h^{ab} may be used to define the square of any form and of any space-like vector but not of time-like vectors. According to what was said previously

$$\nabla_c h^{ab} = 0$$

The remaining information contained in the first law of dynamics may be expressed by

$$t_{[e}R^a{}_{b]cd} = 0$$

and

$$h^{ad}R^b{}_{cde} + h^{bd}R^a{}_{edc} = 0$$

where $R^a{}_{bcd}$ is the curvature tensor.

Pre-relativistic electrodynamics introduces a new geometric element, the ether. It may be defined as a rigging of the hypersurfaces of constant absolute time. Given an ether on N, let u^a be the vector field tangent to the directions of rigging and normalized so that

$$u^a t_a = 1$$

and let us introduce the tensor

$$g^{ab} = h^{ab} - u^a u^b/c^2 \qquad [1]$$

where c is the velocity of light. If u^a is covariantly constant,

$$\nabla_a u^b = 0$$

then

$$\partial_{[a}F_{bc]} = 0 \quad \text{where } F_{ab} = F_{[ab]}$$

and

$$\nabla_b F^{ab} = 0 \quad \text{where } F^{ab} = g^{ac} g^{bd} F_{cd}$$

are equivalent to usual Maxwell's equations for the vacuum.

Clearly, the connection Γ^a_{bc} is metric relative to g^{ab}

$$\nabla_c g^{ab} = 0$$

Moreover, the matrix g^{ab} is non-singular. Its inverse, g_{ab}, together with Γ^a_{bc}, defines a flat indefinite Riemannian (Minkowskian) geometry in N. In pre-relativistic electrodynamics this geometry co-existed with the Newtonian structure; it has been used to define the Lorentz group.

The essential step taken by Einstein in 1905 consisted in denying any physical significance to the Newtonian structure (t, h^{ab}). In special relativity, the geometry of space-time is fully determined by the Minkowski elements (g_{ab}, Γ^a_{bc}). Accordingly, all equations of physics may contain only these elements, in addition to quantities describing the state of the system (this statement is often called the "principle of relativity").

When one attempts to apply Newtonian mechanics in cosmology, one encounters the following apparent difficulty: assume that the Universe is spatially homogeneous and let $\varrho(t)$ be the mean density of matter. A typical solution of Poisson's equation is

$$\varphi = \frac{2}{3} \pi k \varrho r^2$$

The corresponding gravitational field, $-\mathrm{grad}\varphi$, seems to contradict the cosmological principle: the particle at $r = 0$ is unaccelerated while all others are. This difficulty disappears if it is remembered that, in this case, it is impossible to introduce a preferred set of inertial frames defined up to Galilean transformations. The set of all inertial frames is essentially larger and for every galaxy there is one such frame with respect to which the galaxy is at rest.

The assumption of homogeneity and isotropy leads to the following expression for the velocity field of substratum, referred to a certain inertial frame

$$\mathbf{v} = \mathbf{r}\, R^{-1}\, dR/dt \qquad [2]$$

where R is an arbitrary function of the absolute time. The motion of the substratum provides a natural choice for the ether: the rigging is defined by the tangents to the world-lines of elements of the substratum. As can be easily shown, this assumption leads to an expression for the Doppler shift of light coming from distant galaxies, which is identical with the corresponding expression obtained in relativistic cosmology. It is not hard to understand the origin of this coincidence. For a g^{ab} of the form [1], a straightforward calculation gives

$$g_{ab}\, dx^a\, dx^b = d\mathbf{r}^2 - 2\mathbf{v}\cdot d\mathbf{r}\, dt - (c^2 - v^2)\, dt^2 \qquad [3]$$

where

$$x^a = (\mathbf{r}, t),\ \ d\mathbf{r}^2 = dx^2 + dy^2 + dz^2,\ \text{etc.}$$

and a simple coordinate transformation reduces [3], with $\mathbf{v}$ of the form [2], to a Friedmann line-element,

$$R^2\, d\mathbf{r}'^2 - c^2\, dt^2$$

In addition to giving the same formula for the Doppler shift, Newtonian and relativistic cosmologies lead to similar equations for the expansion function $R(t)$. This interesting fact was noticed for the first time by Milne and McCrea in 1934. The following lines contain a brief analysis of the problem: what are the physical situations for which the Newtonian and relativistic descriptions are as close as they are in cosmology?

Let $\mathbf{v}(\mathbf{r}, t)$ be a (sufficiently regular) Newtonian velocity field and $\mathbf{r} = \mathbf{F}(\mathbf{r}', t)$ a family of solutions of

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}(\mathbf{r}, t) \qquad [4]$$

satisfying some initial conditions, say, $\mathbf{F}(\mathbf{r}', 0) = \mathbf{r}'$. The coordinate transformation $\mathbf{r} \to \mathbf{r}'$, with t unchanged, reduces the line-element [3] to

$$\frac{\partial \mathbf{F}}{\partial x^{\alpha'}} \cdot \frac{\partial \mathbf{F}}{\partial x^{\beta'}}\, dx^{\alpha'}\, dx^{\beta'} - c^2\, dt^2$$

where $x^{1'}, x^{2'}, x^{3'}$ are the components of $\mathbf{r}'$.

Consider the Einstein field equations with the cosmological term for a dust of density ϱ and four-velocity $u^a/c = -cg^{ab}t_b$

$$R_{ab} - \frac{1}{2} g_{ab} R + \frac{\lambda}{c^2} g_{ab} = -\, 8\pi k \varrho t_a t_b$$

For a metric of the form [3], equation [4] implies

$$\text{curl curl}\, \mathbf{v} = 0$$

For the sake of simplicity, all further considerations will be restricted to irrotational motions,

$$\text{curl}\, \mathbf{v} = 0$$

The strain tensor may then be written as

$$\partial_\alpha \mathbf{v}_\beta = \frac{1}{3} \delta_{\alpha\beta} \Theta + \sigma_{\alpha\beta}$$

where

$$\Theta = \text{div}\, \mathbf{v}$$

gives the rate of expansion and $\sigma_{\alpha\beta}$ describes the velocity of shear. If we denote $u^a \partial_a \alpha = \partial\alpha/\partial t + \mathbf{v}\cdot\operatorname{grad}\alpha$ by $\dot{\alpha}$, the remaining field equations [4] assume the form

$$\frac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta} - \frac{1}{3}\Theta^2 + \lambda = -8\pi k\varrho \qquad [5]$$

$$2\dot{\Theta} + \Theta^2 + \frac{3}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta} - 3\lambda = 0 \qquad [6]$$

$$\dot{\sigma}_{\alpha\beta} + \Theta\sigma_{\alpha\beta} = 0$$

They imply the equation of continuity,

$$\dot{\varrho} + \varrho\Theta = 0 \qquad [7]$$

On the other hand, the Newtonian equations with a cosmological term,

$$\dot{\mathbf{v}} = -\operatorname{grad}\varphi$$

$$\Delta\varphi = 4\pi k\varrho - \lambda$$

$$\frac{\partial\varrho}{\partial t} + \operatorname{div}(\varrho\mathbf{v}) = 0$$

are equivalent to [7] and

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \sigma_{\alpha\beta}\sigma^{\alpha\beta} - \lambda = -4\pi k\varrho \qquad [8]$$

It is seen by inspection that the relativistic equations [5] and [6] imply the Newtonian [8]. Therefore, to any metric [3], solution of Einstein's equations, with irrotational $\mathbf{v}$, there corresponds an analogous solution of Newton's equations, the functions $\mathbf{v}$ being the same in both cases.

As examples of solutions of the Newtonian equations which lead to Einstein spaces, we mention the following:

1) consider a system of test particles ($\varrho = 0$) falling radially towards the centre of a spherically symmetric body of mass m. If the velocities of the particles vanish at infinity, then, according to Newtonian mechanics,

$$\mathbf{v} = -\sqrt{\frac{2\mathrm{km}}{r}}\,\frac{\mathbf{r}}{r}$$

Substituting this into [3], we obtain the Schwarzschild line-element.

2) in a Newtonian world with a cosmic repulsive force ($\lambda > 0$), a possible motion of test particles is given by

$$\mathbf{v} = \sqrt{\lambda/3}\,\mathbf{r}$$

The corresponding Riemannian metric is that of the de Sitter space.

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