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Investment model

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INVESTMENT MODEL

Investment Model

Let us have a firm that needs investments to continue its functioning. We also have \( n \) investment companies offering theirs investment projects. Let \( N_1^0, N_2^0, \ldots, N_n^0 \) be the initial investments of the companies, \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \) – the increase coefficients of \( N_1, N_2, \ldots, N_n \), and \( \gamma_1, \gamma_2, \ldots, \gamma_n \) – the loss coefficients. Let us have the loss at the unit time function \( F(N_1, N_2, \ldots, N_n) \), which turns into zero together with the sum \( N_1 + N_2 + \ldots + N_n = 0 \).

Then we can represent the change \( N_i, i = 1, n \) in short period of time with the following system of differential equations:

\[
\begin{align*}
\frac{dN_1}{dt}(t) &= (\varepsilon_1 - \gamma_1 F(N_1, N_2, \ldots, N_n)) N_1(t) \\
\frac{dN_2}{dt}(t) &= (\varepsilon_2 - \gamma_2 F(N_1, N_2, \ldots, N_n)) N_2(t) \\
\vdots \\
\frac{dN_n}{dt}(t) &= (\varepsilon_n - \gamma_n F(N_1, N_2, \ldots, N_n)) N_n(t)
\end{align*}
\]

(1)

Solving this set of equations we assume, that if \( t = t_0 : \)

\[ N_i(t_0) = N_i^0, i = 1, n. \] 

(2)

Suppose that one of the investment companies is a big firm with long-term experience. Its activities aren’t influenced by the activities of the other companies (the decisions are made independently). Without loss of generality we
might assume it is the first firm. The inner policy and the future prognosis are supposed to be known. Thus the evolution of this company’s activities we may present as continuous function or continuous clutching of functions.

Assume, that

\[ N_i(t) = f(t), \]  

(3)

is a piecewise continuous function of time variable \( t \), defined on the interval \( t \geq 0 \).

The task is to describe the evolution of other companies.

We will solve the system (1) eliminating the loss at the unit time function \( F(N_1, N_2, ..., N_n) \). The solutions are sought as the dependence of the \( i \)-th project on the 1st.

The solution is obtained as a set:

\[
\begin{align*}
\frac{N^t_i(t)}{N^{t_0}_i(t)} &= C \exp\{(\epsilon_i \gamma_2 - \epsilon_2 \gamma_i)t\} \\
\frac{N^t_j(t)}{N^{t_0}_j(t)} &= C \exp\{(\epsilon_j \gamma_n - \epsilon_n \gamma_j)t\}
\end{align*}
\]

where \( C \) – integration constant.

Using the initial assumptions on the initial amount of the investment of every investment company, we obtain:

\[
\begin{align*}
\frac{N^t_i(t)}{N^{t_0}_i(t)} &= \left(\frac{N^t_i(t)}{N^{t_0}_i(t)}\right) \exp\{(\epsilon_i \gamma_2 - \epsilon_2 \gamma_i)(t - t_0)\} \\
\frac{N^t_j(t)}{N^{t_0}_j(t)} &= \left(\frac{N^t_j(t)}{N^{t_0}_j(t)}\right) \exp\{(\epsilon_j \gamma_n - \epsilon_n \gamma_j)(t - t_0)\}
\end{align*}
\]

(4)

Let’s find the form of evolution equation for \( N_i(t) \), we get:

\[
N_i(t) = \left(\frac{N_i(t)}{N^0_i}\right)^\chi_i N^0_i \exp\{(\epsilon_i - \epsilon_2 \gamma_2)(t - t_0)\}.
\]

Using the assumption about the appearance of the function \( N_i(t) \), we get:

\[
N_i(t) = \left(\frac{f(t)}{N^0_i}\right)^\chi_i N^0_i \exp\{(\epsilon_i - \epsilon_2 \gamma_2)(t - t_0)\} \quad i = 2, n.
\]

(5)
Hence the solutions of the system of differential equations with the initial assumptions (2) and (3) turns into:

\[
\begin{align*}
N_i(t) &= f(t) \\
N_i(t) &= \left(\frac{f(t)}{N^0_i}\right)^\lambda N^0_i \exp\{(e_i - e_i \gamma_i)(t - t_i)\}, i = 2, n.
\end{align*}
\]

Knowing these functions we may find terms of repayment for every investment project. All we need is to solve the following system:

\[
\begin{align*}
N^0_0 &= f(t^*_i) \\
N^0_i &= \left(\frac{f(t^*_i)}{N^0_i}\right)^\lambda N^0_i \exp\{(e_i - e_i \gamma_i)(t^*_i - t_i)\}, i = 2, n.
\end{align*}
\]

simplified:

\[
\begin{align*}
N^0_0 &= f(t^*_i) \\
\ln f(t^*_i) &= \ln N^0_i + (e_i - e_i \gamma_i)(t^*_i - t_i), i = 2, n.
\end{align*}
\]

Among all the solutions we choose the smallest non-negative. Then vector \((t^*_1, t^*_2, ..., t^*_n)\), where \(t^*_i, i = 1, n\) - terms of repayment of every project, solves the system. If some of \(t^*_i, i = 1, n\) equals zero, the project is considered irrepayable and we shall not pay any attention to it.

Let us cast aside nearly impossible case, when \(e_i \gamma_j - e_j \gamma_i = 0\), and assume that \(e_i \gamma_j - e_j \gamma_i > 0\), i.e. \(\frac{e_i}{\gamma_i} > \frac{e_j}{\gamma_j}\), then in accordance with (4):

\[
\lim_{t\to\infty} \frac{N^0_j}{N^0_i} = +\infty.
\]

Since \(N_i\) is bounded, \(N_j\) approaches zero.

So the preference should be given to the investment company which index \(e_j/\gamma\) is the greatest.

Compute the expected profitability of every project. To do this we compute annual profit using evolutionary equations (6).
Assign \( W_i(\tau) = N_i(\tau) - N_i(\tau - 1), \ \tau = 1, T, \ \tau, T \in Z \) – profit functions of \( i \)-th project in \( \tau \)-th year. Then the profitability in this year is
\[
r_i(\tau) = \frac{W_i(\tau) - W_i(\tau - 1)}{W_i(\tau - 1)}, \ \tau = 2, T.
\]
The expected profitability of the project is:
\[
r_i = \frac{1}{T} \sum_{\tau=2}^{T} r_i(\tau).
\]

**Stochastic generalization of the model**

Suppose that the above described model is influenced by the change of the environment. Let us have \( m \) outer factors \( \{A_1, A_2, ..., A_m\} \) forming divisible group of pairwise independent events;
\[
P(A_1), P(A_2), ..., P(A_m) \quad \text{— their probabilities. The following equality holds:}
\sum_{i=1}^{m} P(A_i) = 1.
\]

Write the system of differential equations, similarly to (1), describing the change of \( N_i \) in short time period taking into account the influence of the environment. Denote \( N_i(A_k) \) – cash flow of the \( i \)-th investment project on condition of the influence of \( k \)-th factor. We should mention that the loss and profit coefficients now depend on the state of environment. Denote them \( \epsilon_i(A_k), \gamma_i(A_k), i = 1, n, k = 1, m \). Assume that the capital inflows of every investment company do not depend on the state of the environment (it is the amount of money defined in advance). Then the system will look like:

\[
\begin{align*}
\frac{dN_1(t, A_1)}{dt} &= (\epsilon_1(A_1) - \gamma_1(A_1)F(N_1, N_2, ..., N_n))N_1(t, A_1) \\
\frac{dN_1(t, A_2)}{dt} &= (\epsilon_1(A_2) - \gamma_1(A_2)F(N_1, N_2, ..., N_n))N_1(t, A_2) \\
&\vdots \\
\frac{dN_1(t, A_m)}{dt} &= (\epsilon_1(A_m) - \gamma_1(A_m)F(N_1, N_2, ..., N_n))N_1(t, A_m) \\
\frac{dN_2(t, A_1)}{dt} &= (\epsilon_2(A_1) - \gamma_2(A_1)F(N_1, N_2, ..., N_n))N_2(t, A_1) \\
\frac{dN_2(t, A_2)}{dt} &= (\epsilon_2(A_2) - \gamma_2(A_2)F(N_1, N_2, ..., N_n))N_2(t, A_2) \\
&\vdots \\
\frac{dN_1(t, A_m)}{dt} &= (\epsilon_2(A_m) - \gamma_2(A_m)F(N_1, N_2, ..., N_n))N_2(t, A_m)
\end{align*}
\]
and the initial conditions:

\[ N_i(t_0, A_k) = N_0^i, \quad i = 1, n, \quad k = 1, m. \]  \hspace{1cm} (12)

Like in previous case, we solve this system, using the initial conditions, and the assumption about the 1st company’s functioning, which evolution function now turns into:

\[ N_i(t, A_k) = f(t, A_k), \quad k = 1, m. \]  \hspace{1cm} (13)

The solution:

\[
\begin{aligned}
N_i(t, A_k) &= f(t, A_k) \\
N_i(t, A_k) &= \left( \frac{f(t, A_k)}{N_i^0} \right)^{1/(\gamma_i(A_k))} N_i^0 \exp\{ (\varepsilon_i(A_k) - \varepsilon_i(A_k) \gamma_i(A_k)) (t - t_i) \}, \quad i = 2, n.
\end{aligned}
\]  \hspace{1cm} (14)

The selection of the best investor now depends on the state of environment.

So, having the predicted indexes \( \varepsilon_i(A_k), \gamma_i(A_k), \quad i = 1, n, \quad k = 1, m, \) we can define most attractive project.

Suppose that the case when \( \varepsilon_i(A_k) \gamma_i'(A_k) - \varepsilon_j(A_k) \gamma_j'(A_k) = 0, \quad k = 1, m \) is impossible, and \( \varepsilon_i(A_k) \gamma_i'(A_k) - \varepsilon_j(A_k) \gamma_j'(A_k) > 0, \) which means \( \frac{\varepsilon_i(A_k)}{\gamma_i'(A_k)} > \frac{\varepsilon_j(A_k)}{\gamma_j'(A_k)}, \quad k = 1, m, \) then in response to (14) we have

\[
\lim_{i \to \infty} N_i^j(A_k) = +\infty, \quad k = 1, m. \]  \hspace{1cm} (15)

As \( N_i \) is bounded, then \( N_j \) follows zero.

So when the \( k \)-th factor influences, we should prefer the investment company which index \( \frac{\varepsilon(A_k)}{\gamma(A_k)} \) is the largest.
Now we will realize the outer factor averaging.

\[
N_i(t) = \sum_{k=1}^{n} P(A_k) f(t, A_k) \tag{16}
\]

\[
N_i(t) = \sum_{k=1}^{n} P(A_k) \left( \frac{f(t, A_k)}{N_i} \right)^{\frac{1}{\gamma_i(A_k)}} N_i^\gamma \exp\{(\epsilon_i(A_k) - \bar{\epsilon}_i)(A_k) - \gamma_i(A_k)(t-1)\}, i = 2, n.
\]

Denote

\[
\bar{\gamma}_i = \sum_{k=1}^{n} P(A_k) \gamma_i(A_k), \quad i = 1, n. \tag{17}
\]

Substitute the coefficients of increases and losses in (1) for their average indexes.

\[
\begin{align*}
\frac{dN_1(t)}{dt} &= (\epsilon_1 - \bar{\gamma}_1 F(N_1, N_2, ..., N_n))N_1(t) \\
\frac{dN_2(t)}{dt} &= (\epsilon_2 - \bar{\gamma}_2 F(N_1, N_2, ..., N_n))N_2(t) \\
&\quad \text{(for } i = 2, n) \\
\frac{dN_n(t)}{dt} &= (\epsilon_n - \bar{\gamma}_n F(N_1, N_2, ..., N_n))N_n(t)
\end{align*} \tag{18}
\]

Solving this system with the initial conditions (2) and assumption (3), we obtain:

\[
\begin{align*}
N_1(t) &= \gamma_1(t) \\
N_i(t) &= \left( \frac{\Gamma(t)}{N_i} \right)^{\frac{1}{\gamma_i}} N_i^\gamma \exp\{(\epsilon_i - \bar{\epsilon}_i)(A_k)(t-1)\}, i = 2, n.
\end{align*} \tag{19}
\]

The natural question is arising: when the average solutions converge to the solutions of averaged system? Graphical analysis of the concrete cases showed that only the first project’s function given in system (16) converges to the respective function of the system (19). For the other projects there is no convergence. One might suppose that this convergence is possible under some conditions on the increase coefficients \( \epsilon_1, \epsilon_2, ..., \epsilon_n \) and loss coefficients \( \gamma_1, \gamma_2, ..., \gamma_n \).

Theoretical provement of this convergence remains open.

Similarly to the previous case, one may compute expected profitability of every project. To do it we compute annual incomes for every outer factor.

Let \( W_i(\tau, A_k) = N_i(\tau, A_k) - N_i(\tau-1, A_k), \quad \tau = 1, T, \quad \tau, T \in \mathbb{Z}, k = 1, m \) be the profit of \( i \)-th project in \( k \)-th year within the state of the environment \( A_k \). Then
the profitability is 
\[ r_i(\tau, A_k) = \frac{W_i(\tau, A_k) - W_i(\tau - 1, A_k)}{W_i(\tau - 1, A_k)} , \quad \tau = 2, T \]

and the expected profitability of the project under conditions of the affect of k-th factor:
\[ r_i(A_k) = \frac{1}{T} \sum_{\tau=2}^{T} r_i(\tau, A_k) . \tag{20} \]

We should mark here, that one may compute the expected profitability of the project using this formulas and taking into account that among the numbers \( r_i(\tau, A_k) \) there are both coefficients of profitability and loss.

Using the developed model

*Example 1.*

Consider a simple case, when the evolution of the investment projects is not being affected by the environment. Let us have 10 companies offering their money. The task is to find the most attractive project, even if it is known there is at least one reliable investor among them with good reputation and well thought-out investment policy. First we suppose the amount of capital inflow of every investor is equal. We know indexes \( e_i \) and \( \gamma_i \), \( i = 1,10 \) predicted by the projects.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
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<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<tr>
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<td>1500000</td>
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<td>1500000</td>
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<td>1500000</td>
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<td>1500000</td>
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<tr>
<td>( e_i )</td>
<td>0.38</td>
<td>0.28</td>
<td>0.38</td>
<td>0.42</td>
<td>0.3</td>
<td>0.45</td>
<td>0.44</td>
<td>0.5</td>
<td>0.34</td>
<td>0.41</td>
</tr>
<tr>
<td>( \gamma_i )</td>
<td>0.89</td>
<td>0.85</td>
<td>0.98</td>
<td>0.8</td>
<td>0.92</td>
<td>0.89</td>
<td>0.97</td>
<td>0.95</td>
<td>0.99</td>
<td>0.85</td>
</tr>
</tbody>
</table>

Suppose the evolution of the first investment project is given by the function.

\[ N_i(t) = \begin{cases} N_1^i(1 + e, t - \gamma, t) + \frac{(e, t)^2}{2}, t \leq t_0 \\ N_2^i(-1 + \gamma, t) + N_0^i(2 + e, t - 2\gamma, t) + \frac{(e, t)^2}{2}, t \geq t_0 \end{cases} \]

and \( t_0 = 7 \) years. For further calculations and plot construction we will use package *Mathematica*. Graphically evolution can be shown:
The equation for computing the repayment period of the investment was deduced earlier. We will use the first one. Obtain that this project will repay itself already in 7,036 years.

Substitute the function \( N_i(t) \) into equations for other projects:

\[
N_i(t) = \begin{cases} 
(1 + e_i t - \gamma_i t + t (e_i t)^2 \frac{\gamma_i}{2} N_i^* \exp \{ (e_i - e_i \gamma_i t) t \}, t \leq t_0 \\
(1 + \gamma_i t + e_i t + 2 \gamma_i t + (e_i t)^2 \frac{\gamma_i}{2} N_i^* \exp \{ (e_i - e_i \gamma_i t) t \}, t \geq t_0
\end{cases}
\]

for \( i = 2, 10 \). Construct plots for the rest 9 projects using data from Table 1.

Project 2: \( e_2 = 0.28, \gamma_2 = 0.85 \).

Such project repay itself in 8,13 years, but it is obvious from the plot that there comes the moment, when its profitability starts falling; the project becomes unprofitable.

Project 3: \( e_3 = 0.38, \gamma_3 = 0.98 \).
Fig. 3. Evolution in time of $N_3(t)$.
Repayment period: 7.27 years.
Project 4: $e_4 = 0.42$, $g_4 = 0.8$.

Fig. 4. Evolution in time of $N_4(t)$.
Repayment period: 6.547 years.
Project 5: $e_5 = 0.3$, $g_5 = 0.92$.

Fig. 5. Evolution in time of $N_5(t)$.
Repayment period: 8.274 years. Profits of such project decrease in time and there comes the moment when it becomes unprofitable.
Project 6: $e_6 = 0.45$, $g_6 = 0.89$. 
Fig. 6. Evolution in time of $N_6(t)$.
Repayment period: 6,619 years.
Project 8. $\varepsilon_7 = 0.44$, $\gamma_7 = 0.97$.

Fig. 7. Evolution in time of $N_7(t)$.
Repayment period: 6,867 years.
Project 8. $\varepsilon_8 = 0.5$, $\gamma_8 = 0.95$.

Fig. 8. Evolution in time of $N_8(t)$.
Repayment period: 6,543 years.
Project 9. $\varepsilon_9 = 0.34$, $\gamma_9 = 0.99$. 
Fig. 9. Evolution in time of $N_9(t)$. Repayment period: 7.94 years. Profits of such project decrease in time and there comes the moment when it becomes unprofitable.

Project 10: $e_{10}^1 = 0.41$, $g_{10}^1 = 0.85$.

Fig. 10. Evolution in time of $N_{10}(t)$. Repayment period: 6.719 years.

Let’s form a table containing index $e_i/g_i$ and repayment period $t_i^*$, $i = 1,10$.

<table>
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<th>7</th>
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</tr>
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<tr>
<td>$e_i/g_i$</td>
<td>0.427</td>
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<td>0.388</td>
<td>0.525</td>
<td>0.326</td>
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<td>0.526</td>
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It was shown that we should prefer the project with the largest $e_i/g_i$. That is why in our case we should prefer 4th and 8th projects. Moreover one can see the repayment periods of these projects are the smallest and almost equal. Find the expected profitability of every project. To do this we form a table of values of $N_i(k)$, $i = 1,10$, $k = 1,25$ (Table 3). Then we compute annual profit for every project (Table 4) and coefficients of profitability (Table 5).
Table 3. Evolution $N_i(t)$ in years

<table>
<thead>
<tr>
<th>year</th>
<th>1</th>
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<th>3</th>
<th>4</th>
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</table>

Using formula (9) compute expected profitability of every project (Table 6). We obtained that only 2nd, 4th, 8th and 10th projects are profitable during first 25 years.

However graphical analysis has shown us that we should throw away 2nd project if we need a long-term project.
Table 4. Profit of every project (yearly).

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</table>

**Example 2.**

Now consider a case with the influence of environment on the investment project. Let it be given with the complete set of pairwise independent events \( \{ A_1, A_2, ..., A_n \} \) with probabilities \( P(A_k) = \frac{1}{6} \). We have 3 companies offering their investments. Suppose the amount of capital inflows of every investor is equal regardless of the conditions.
Table 5. Coefficients of profitability for every project (yearly).

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Table 6. Expected profitability the project.

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<th>4</th>
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We know indexes $e_i(A_k)$ and $\gamma_i(A_k), \ i=1,3, \ k=1,6$ predicted by the projects.

Preliminary data are shown in the table:
Table 7.

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<th>6</th>
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<td>0.44</td>
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<td>0.84</td>
<td>0.9</td>
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<td>0.92</td>
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</tr>
</tbody>
</table>

Let $N_i^0 = 1500000$ monetary units, $i=1,3$. Similarly to the case when there is no outer influence suppose that the evolution function for the first investment project is known and given by:

$$
N_i(t,A_k) = \begin{cases} 
N_i^0(t) = N_i^0(1 + \varepsilon_i(A_k)t - \gamma_i(A_k)t + \frac{\varepsilon_i(A_k)t^2}{2}), & t \leq t_0 \\
N_i^0(-1 + \gamma_i(A_k)t) + N_i^0(2 + \varepsilon_i(A_k)t - 2\gamma_i(A_k)t + \frac{\varepsilon_i(A_k)t^2}{2}), & t > t_0
\end{cases}
$$

where $k=1,6$ and $t_0 = 7$.

We are using Mathematica for computations and to build plots of functions. Find the averaged solution for the first project:

$$
\overline{N_i}(t) = \begin{cases} 
1500 - 721.5t + 130.29t^2, & t \leq 7 \\
-6532 + 1338t^2, & t > 7
\end{cases}
$$

The plot of this function is Fig. 11.

Fig. 11. Evolution in time of $\overline{N_i}(t)$

Substitute $N_i(t,A_k)$ into the solution for second and third project and find their averagings:

$$
\overline{N}_2(t) = 18e^{0.014}(t-4)(t-3.13) + 13.5e^{0.121}(t-1.1)(t-1.87) +
+ 150e^{-0.389}(1-0.39t+0.07t^2)^{1.12} + 300e^{0.003}(1-0.38t+0.09t^2)^{1.15} +
+ 150e^{0.051}(1-0.44t+0.1t^2)^{1.08} + 375e^{-0.041}(1-0.45t+0.125t^2)^{0.05}, & t \leq 7
+ 150e^{-0.111}(-3.7 + 0.8t)^{1.12} + 300e^{0.003}(-3 + 0.8t)^{1.15} +
+ 150e^{0.031}(-3.35 + 0.89t)^{1.08} + 300e^{0.13}(-7.6 + 0.92t) +
+ 375e^{-0.041}(-2.7 + 0.95t)^{0.08} + 225e^{0.041}(-5.86 + 0.97t)^{2}, & t \geq 7
$$
The plot of this function looks like:

Fig. 12. Evolution in time of $\tilde{N}_2(t)$

$$\tilde{N}_2(t) = \begin{cases} 
18e^{-144}(t-4)(t-3.125) + 13.5e^{-140}(t-11.92)(t-1.87) + \\
+150e^{-121}(1-0.39t + 0.07t^2) + 300e^{-123}(1-0.38t + 0.09t^2) + \\
+150e^{-0.121}(1-0.44t + 0.1t^2) + 375e^{-0.131}(1-0.45t + 0.125t^2), t \leq 7 \\
150e^{-0.642}(-3.7 + 0.76t)^{2.4} + 300e^{-0.824}(0.8t - 2.94) + \\
150e^{-0.111}(-3.35 + 0.89t)^{2.0} + 300e^{-0.123}(0.92t - 7.575) + \\
+375e^{-0.121}(0.95t - 2.675)^{0.92} + 225e^{-0.140}(0.97t - 5.86), t \geq 7 
\end{cases}$$

The plot of this function looks like:

Fig. 13. Evolution in time of $\tilde{N}_3(t)$

Fig. 14. Evolution in time of the averaged solutions
One can easily see that with such data, basing our decision upon the averaging, we should prefer the first project for short-term projects (to 10 years) and third one for the investment projects lasting over than 10 years.

Compute the index $\frac{e_i(A_k)}{\gamma_i(A_k)}$ for $i = 1, 3, k = 1, 6$. Obtained results summarized in the table 8. One can see that in every state of the environment there exists most attractive project (in states 1 and 4 – second, in states 2, 3, 5 – third, in the state 6 – first).

Using formulas find the averagings for the coefficients of increase and expenses.

$\bar{\varepsilon}_1 = 0.411, \bar{\varepsilon}_2 = 0.4375, \bar{\varepsilon}_3 = 0.3965, \bar{\gamma}_1 = 0.892, \bar{\gamma}_2 = 0.9085, \bar{\gamma}_3 = 0.891.$

Table 8.

<table>
<thead>
<tr>
<th>$k$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon_i(A_k) \gamma_i(A_k)$</td>
<td>0.506</td>
<td>0.326</td>
<td>0.412</td>
<td>0.525</td>
<td>0.487</td>
<td>0.526</td>
</tr>
<tr>
<td>$\varepsilon_i(A_k) \gamma_i(A_k)$</td>
<td>0.556</td>
<td>0.467</td>
<td>0.454</td>
<td>0.556</td>
<td>0.333</td>
<td>0.478</td>
</tr>
<tr>
<td>$\varepsilon_i(A_k) \gamma_i(A_k)$</td>
<td>0.389</td>
<td>0.489</td>
<td>0.557</td>
<td>0.375</td>
<td>0.511</td>
<td>0.379</td>
</tr>
</tbody>
</table>

Find the solutions of the averaged system (denote them as $\hat{N}_i(t)$):

$\hat{N}_1(t) = \begin{cases} 1500 - 721.5t + 126.69t^2, & t \leq 7 \\ -6708.65 + 1338t, & t \geq 7 \end{cases}$

$\hat{N}_2(t) = \begin{cases} 1500e^{0.101t}(1 - 0.481t + 0.0845t^2)^{0.999}, & t \leq 7 \\ 1500e^{0.101t}(-4.473 + 0.892t)^{0.999}, & t \geq 7 \end{cases}$

$\hat{N}_3(t) = \begin{cases} 1500e^{-0.014t}(1 - 0.481t + 0.0845t^2)^{0.999}, & t \leq 7 \\ 1500e^{-0.014t}(-4.473 + 0.892t)^{0.999}, & t \geq 7 \end{cases}$

Fig. 15. Evolution in time of $\hat{N}_1(t)$ and $\hat{N}_3(t)$.
It is easy to see that the averaged solutions converge to the solutions of the averaged system only for the first project; for other projects this converges needs additional conditions on the coefficients of increase and expenses.

**SUMMARY**

The article presents certain model of choice of investment projects by firm in deterministic and stochastic approach. It is based on system of differential equation. The solution gives tool in decision process. Two numerical implementation of model are given.

**MODEL WYBORU PROJEKTÓW INWESTYCYJNYCH**

**STRESZCZENIE**

W artykule przedstawiono pewien model wyboru projektów inwestycyjnych przez firmę w wariancie deterministycznym i stochastycznym. Właściwym modelem tworzy układ równań różniczkowych zwyczajnych. Jego rozwiązanie pozwala na przyjęcie
odpowiedniej decyzji. Zaprezentowano dwa warianty implementacji numerycznej modelu.

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