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THE EXISTENCE OF SINGULARITIES AND THE ORIGIN OF SPACE-TIME

Methods of noncommutative geometry are applied to deal with singular space-times in general relativity. Such space-times are modeled by noncommutative von Neumann algebras of random operators. Even the strongest singularities turn out to be probabilistically irrelevant. Only when one goes to the usual (commutative) regime, via a suitable transition process, space-time emerges and singularities become significant.

1. INTRODUCTION

There is a general feeling that the future theory of quantum gravity will eliminate singularities from the physical image of the world. This widespread opinion contributes to the fact that to the majority of cosmologists and astrophysicists the singularity problem seems to be now less important than it seemed to be several years ago. It is tacitly assumed that there are only two possibilities: either the “final theory” will be singularity free, or not, and the latter possibility is less and less popular. However, the history of physics teaches us that a truly generalized theory can contain, as special cases, possibilities that previously seemed to be mutually exclusive. I will show that this is indeed the case when the singularity problem is treated in a truly generalized setting.

In the classical approach, singularities were regarded as elements of space-time boundaries rather than as “internal points” of space-time [2, 5, 15]. Various space-time boundaries were constructed (g-boundary, b-boundary, conformal and causal boundaries...) with the view of taming singularities, i.e., making them tractable from the mathematical point of view. An alternative approach could be to look for new mathematical tools and to face the problem in its full mathematical complexity. It is noncommutative geometry that has been created with the view of dealing with “highly singular” situations in mathematics [3], and in this note I will apply its methods to meet the challenge of strong singularities in cosmology (for technical details see [7]).

In Section 2, some key mathematical tools to deal with singularities are reminded, and with their help the truly malicious character of strong singularities is displayed. A noncommutative method, as it is applied to the singularity problem, is briefly presented in Section 3, and this method is put to work in Section 4. It is shown that space-time with strong (the so-called malicious) singularities is modeled by a von Neumann algebra of random operators which makes singularities probabilistically irrelevant. A physical interpretation of this result is sketched in Section 5.

2. MALICIOUS SINGULARITIES

There are reasons to believe that it is Schmidt’s b-boundary construction [14] and its later history that best reveal the complex mathematical structure of strong curvature singularities. We first briefly recall this construction. Let us consider a space-time (M, g), and $\pi_M : E \to M$ a fibre bundle of linear frames over M with the Lorentz group $G = SO(3, 1)$ as its structural group. Levi-Civita connection on M determines the family of Riemann metrics on E. We select one of them, and notice that the further construction does not depend on this choice. With the help of the chosen metric we determine the distance function on E and construct the Cauchy completion $\bar{E}$ of E. It can be shown that the right action of the group G on E prolongs to that of $\bar{E}$. This allows us to define the quotient space $\bar{M} = \bar{E}/G$. We call $\bar{M}$ the *b-completion* of space-time M. M is open and dense in $\bar{M}$. Finally, we define the *b-boundary* of space-time M as $\partial_b M = \bar{M} - M$. Each g-incomplete geodesic and each incomplete timelike curve of bounded acceleration (that can be interpreted as the history of a physical body) in (M, g) determines a point in $\partial_b M$.

After its publishing, Schmidt's construction was regarded "elegant" [5, p. 276] and "most promising" [15, p. 152]. However, when Bosshard [1] and Johnson [12] computed b-boundaries for the closed Friedman world model and for the Schwarzschild solution, it turned out that the b-boundaries of these two space-times consisted of a single point that was not Hausdorff separated from the rest of space-time. This was especially disastrous for the Friedman closed model since this meant that the initial and final singularities form (topologically) the same boundary point. There were several attempts to cure the situation, but none of them was fully successful (for a review see [4]), and Schmidt's construction went slowly in oblivion.

However, instead of rejecting Schmidt's construction it is interesting to clarify the situation. This can be done with the help of slightly generalized but otherwise standard methods of differential geometry and topology [11]. It turns out that it is an equivalence relation $\rho \subset E \times E$, defined by $p_1 \rho p_2 \Leftrightarrow$ there exists $g \in G$ such that $p_2 = p_1 g$, that is responsible for the pathological behavior. Let x_0 be a b-boundary point, and $p_0 \in \pi_{\bar{M}}^{-1}$. We say that the singularity remains in a close contact with equivalence classes $[p]$ of all $p \in E$, if $p_0 \in \mathrm{cl}[p]$ for every $p \in E$. If this is the case, the singularity is called *malicious*. We have demonstrated that both the initial and final singularities in the closed Friedman model, and the central singularity in the Schawrzschild solution are malicious singularities. In this case, the fiber over the singularity reduces to a single point [11]. This is why Schmidt's construction does not work well.

3. NONCOMMUTATIVE APPROACH

To translate Schmidt's construction into the noncommutative language we apply the standard method of making a space noncommutative [3, pp. 86–87]. We consider the fibre bundle $\pi_M\colon E \to M$, as above. The structural group G acts on E, $E \times G \to E$. This allows us naturally to define the groupoid $\Gamma = E \times G$ which is isomorphic to $\bigcup_x E_x \times E_x$ where E_x is the fiber over $x \in M$. We further define the noncommutative algebra $\mathcal{A} = C_c^\infty(\Gamma, \mathbf{C})$ of smooth, compactly supported, complex valued functions on Γ with the convolution as multiplication [8, 13]

$$(a_1 * a_2)(p_1, p_2) = \int_{E_x} a_1(p_1, p_3) a_2(p_3, p_2)\, dp_3$$

for $a \in \mathcal{A}$, $p_1, p_2, p_3 \in E_x$, $x = \pi_M(p_3)$. The algebra $\mathcal{A}$ has the vanishing center, but we define the "outer center" of this algebra as $Z = \pi_M^*(C^\infty(M))$ which acts on the algebra $\mathcal{A}$, $\alpha\colon Z \times \mathcal{A} \to \mathcal{A}$, in the following way

$$\alpha(f, a)(p_1, p_2) = f(p_1) a(p_1, p_2),$$

$a \in \mathcal{A}$, $f \in Z$. It is clear that the geometry of space-time M is encoded in Z (in [7] we have shown that Z can be restricted to bounded functions on M with no loss of generality).

Let us now define $\bar{E} = E \cup \{p_0\}$ and assume that the point p_0 remains in close contact with all equivalence classes of the relation ρ. We extend this relation to $\bar{\rho} \subset \bar{E} \times \bar{E}$ by assuming $(p_0, p_0) \in \bar{\rho}$. Let us denote $\bar{E}/\bar{\rho}$ by $\bar{M}$. We now have the groupoid $\bar{\Gamma} = \Gamma \cup \{(p_0, p_0)\}$ over $\bar{E}$.

We now construct the algebra $\bar{\mathcal{A}}$ on the groupoid $\bar{\Gamma}$. We first define the algebra $\mathcal{A} \oplus Z$ with the multiplication

$$(a + f) * (b + g) = a * b + \alpha(f, b) + \alpha(g, a) + f \cdot g,$$

then we extend this algebra to $\bar{\Gamma}$ (functions from $\mathcal{A}$ are extended through zero, i.e., we put $\bar{a}(p_0, p_0) = 0$, and $\bar{a}|\Gamma = a$). Finally, we set $\bar{\mathcal{A}} = \mathcal{A} \oplus \bar{Z}$.

In the presence of a malicious singularity the algebra Z reduces to $\bar{Z}$ consisting only of constant functions. This corresponds to the fact that space-time M with at least one malicious singularity collapses to a single point. However, in agreement with the philosophy of noncommutative geometry, all information about space-time with malicious singularities is stored in the algebra $\bar{\mathcal{A}}$.

4. IRRELEVANCE OF MALICIOUS SINGULARITIES

Let us define the representation, the so-called regular representation, of the algebra $\mathcal{A}$ in the Hilbert space $\mathcal{H}_p = L^2(\Gamma^p)$ where Γ^p denotes all elements of Γ with p as a second element of the pair. The representation is of the form

$$(\pi_p(a)\psi)(p_1, p) = \int_{E_x} a(p_1, p_2)\psi(p_2, p)\, dp_2$$

for $a \in \mathcal{A}$, $\psi \in \mathcal{H}_p$, $p, p_1, p_2 \in E_x$ with $x = \pi_M(p)$. For $\bar{f} = \lambda =$ const $\in \bar{Z}$ we have

$$(\pi_p(\bar{f})\psi)(p_1, p) = \lambda\psi(p_1, p).$$

It comes as a surprise that every $a \in \mathcal{A}$ generates a random operator $r_a = (\pi_p(a))_{p\in E}$ acting on a collection of Hilbert spaces $\{\mathcal{H}_p\}_{p\in E}$.

An operator valued function $E \ni p \mapsto r(p) \in \mathcal{B}(\mathcal{H})_p$ is called a *random operator* if it satisfies the following conditions [3, pp. 50–53]:

(1) If $\xi_p, \eta_p \in \mathcal{H}_p$ then the function $E \to \mathbf{C}$, given by $E \ni p \mapsto (r(p)\xi_p, \eta_p)$, $a \in \mathcal{A}$, is measurable with respect to the usual manifold measure on E.

(2) The operator r is *bounded*, i.e., $\|r\| < \infty$, $\|r\| = \text{ess sup}\|r(p)\|$, where "ess sup" denotes supremum modulo zero measure sets (the so-called essential supremum).

Operators $r_a = (\pi_p(a))_{p\in E}$, satisfy these conditions.

We define the algebra $\mathcal{M}_0$ of equivalence classes (modulo equality almost everywhere) of random operators $r_a, a \in \mathcal{A}$, and complete it to the von Neumann algebra $\mathcal{M} = \mathcal{M}_0''$ where $\mathcal{M}_0''$ denotes the double commutant of $\mathcal{M}_0$. The algebra $\mathcal{M}$ is called *von Neumann algebra of the groupoid* Γ. Probabilistic aspects of the algebra $\mathcal{M}$ are studied in [9] (it should be remembered that the theory of noncommutative von Neumann algebras is regarded as the measure theory of noncommutative spaces [3, pp. 45–48]).

We repeat the same with the algebra $\bar{\mathcal{A}}$, and define the algebra $\bar{\mathcal{M}}_0$ consisting of random operators of the type $r_{\bar{a}}$ with $\bar{a} \in \bar{\mathcal{A}}$ where $\bar{a} = a + \lambda$. From the von Neumann theorem on the double commutant we have $\mathcal{M}_0'' = \mathcal{M}$. It can be easily seen that also $\bar{\mathcal{M}}_0'' = \mathcal{M}$. This follows from the fact that random operators are equivalence classes of the equivalence relation that is defined by equality “almost everywhere”. As the consequence we have

$$\mathcal{M}_0'' = \bar{\mathcal{M}}_0'' = \mathcal{M}.$$

This is an important equality. Let us notice that $\mathcal{M}_0$ is the algebra of random operators before the singular point has been attached, and $\bar{\mathcal{M}}_0$ is the algebra of random operators after the singular point has been attached. If we complete these two algebras to the von Neumann algebra, we obtain the same von Neumann algebra $\mathcal{M}$. This means that from the probabilistic point of view the singular point is irrelevant.

5. INTERPRETATION

Our main result can be described in the following way. If we decide to encode information about space-time M with at least one malicious singularity in the algebra $\bar{\mathcal{A}} = C_c^\infty((\bar{\Gamma}, \mathbf{C}), *)$ then it turns out that the regular representation of this algebra on a family of Hilbert spaces is an algebra of random operators which makes the initial singularity probabilistically insignificant. This result, obtained in a mathematically rigorous way, suggests an interesting physical interpretation.

Let us notice that, in the present work, singularities were treated in a purely classical way; so far quantum effects were not even mentioned. However, algebras of bounded operators on Hilbert spaces and, in particular, von Neumann algebras, are typically quantum mathematical structures. It looks, therefore, as if malicious singularities "knew something" about the quantum aspect of reality. In fact, basing on this suggestion and on this mathematical structure, a model has been constructed unifying general relativity with quantum mechanics [6, 8, 9, 10]. If we agree that noncommutative geometry will somehow be engaged into the future quantum gravity theory then the problem is not of whether singularities exist on the fundamental level or not, but rather of whether they are relevant or not.

Usually, when the origin of the universe is discussed, two mutually exclusive possibilities are tacitly assumed: either the universe had a singular beginning, or not. However, as we have demonstrated, the third answer is possible: on the fundamental level even malicious singularities are irrelevant. In this approach, probability, albeit in a generalized (noncommutative) sense, is an inherent aspect of the fundamental level, and in this probability dominated regime singularities are irrelevant. Only when one goes to the macroscopic level, space-time appears and the singularities acquire their significance.

This transition can be made mathematically precise. It is clear that the information about space-time is contained in the outer center Z, and to go from the noncommutative regime to the usual space-time geometry one must somehow restrict the algebra $\mathcal{A}$ to Z. In [10] we have shown that this can be done with the help of a suitable "averaging" of elements of $\mathcal{A}$. As we remember, in the presence of a malicious singularity the outer center Z reduces to constant functions, and space-time with its singular boundary collapses to a single point. To avoid this degeneracy one must use sheaf of algebras rather than a single algebra (for details see [11]).

We can speculate that since noncommutative algebras are nonlocal (the concepts of a point and its neighbourhood are meaningless), the fundamental level is aspatial and atemporal (on this level there is no space and no time in the usual sense). The universe simply is, and is immersed in an overwhelming probabilistic aspect. Only from the point of view of the macroscopic observer the question of whether it had a singular beginning in its finite past and will have a singular end in its finite future, acquires its dramatic meaning.

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