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# All the mathematics in the world : logical validity and classical set theory

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# All the mathematics in the world: logical validity and classical set theory

*In memory of Georg Kreisel (1923–2015)*

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Abstract

A recognizable topological model construction shows that any consistent principles of classical set theory, including the validity of the law of the excluded third, together with a standard class theory, do not suffice to demonstrate the general validity of the law of the excluded third. This result calls into question the classical mathematician's ability to offer solid justifications for the logical principles he or she favors.

Keywords

justification of deduction; set validity; class validity; general validity; intuitionism; topological models

## 1. Introduction

In his (1972, 152–153), Kreisel underscored the distinction between set validity and class validity, and pointed out that, under metatheoretic conditions not unusual, the two notions may pull apart extensionally. As will be proved in a moment, it is no paradox to assert that all the classical set-theoretic mathematics in the world, with it the set validity of the unrestricted law of the excluded third [**TND**], plus a class theory with Full Comprehension, if consistent, do not suffice to demonstrate that even the propositional version of the excluded third principle is class valid.

**Definition 1** ([*propositional*] *law of the excluded third* [**TND**]) The *(propositional) law of the excluded third* is the formula

$$p \vee \neg p,$$

for $p$ any propositional variable.

Take any domain of sets the classical mathematician chooses and no matter what (consistent) mathematics he or she discovers for that domain, it will be impossible, putting to work all relevant theorems and techniques, to show that **TND** is valid universally over the domain, even in the presence of a standard class theory. More specifically,

> one can adopt the classical mathematics, if consistent, taken to hold true over any structure $S$ for sets – including **TND** for $S$-and

yet be unable to deduce, from that mathematics plus Full Comprehension for classes, that **TND** governs all the classes over *S*.

Therefore, the presumptive ruling assumptions of classical mathematics cannot convince a latter-day intuitionist that, at work in mathematics, there are precisely two truth-values, **True** and **False**, an assertion tantamount to the validity of the excluded third.

The proof of this fact enlists the aid of a topological model construction in the venerable tradition of Tarski and Scott, for which construction no originality is here claimed. Nor is there any shattering novelty in the theorem proved thereby. However, there is a moral to be extracted from that theorem and from the constructions looming large in its proof, a moral concerning justifications of deduction. The moral in question is twofold. First, it is mistaken to suppose that individual logical principles have some vague manner of epistemic primitivity that prohibits their general validity being proven without begging the question. This is widely believed, at least among philosophers, despite the plain fact that the validity, of either the general, the class, or the set variety is defined in terms of strictly mathematical statements that *prima facie* cry out for proof or disproof. This notion is a close relative to the false idea that there is "nothing to logic," that there is something (deeply?) basic or obvious or (weasel word) 'potentially obvious' (Quine, 1970, p. 82) about the propositional and first-order logical truths, that there is nothing substantial from which the validity of individual truths of logic can be (or even need be) proven, as if the class of solutions to a numerical problem equivalent to the halting problem always has 'obvious' criteria for membership.

The second moral is that the unbiased question of the validity of even propositional logical schemes can be a significant one mathematically. Cognoscenti already know that, when formulated with a little care, the axioms of Zermelo-Fraenkel [**ZF**] set theory do not suffice to prove the set validity of the law of the excluded third, if that validity is not assumed from the outset. However, once the Axiom of Choice, even for varieties of finite sets, is added to **ZF**, a proof of **TND**'s set validity – a proof perfectly adequate from every epistemological standpoint – becomes immediate; it incorporates a charming little argument due to Radu Diaconescu (Diaconescu, 1975), later rediscovered by Goodman and Myhill (Goodman and Myhill, 1978) and (Beeson, 1985, p. 163). Indeed, attempts by classically-minded mathematicians to deduce the validity of excluded third not set-theoretically but generally will fall short – as this essay demonstrates – but the shortfall, I maintain, is neither plain nor superficial. To deduce a validity statement in a non-circular fashion can be a serious mathematical undertaking, and shortfalls in or limits upon such deductions can present substantial mathematical problems, with interesting solutions (McCarty, 2018).

This take on logical validity, a true one, looms large among the insights of the traditional intuitionists L.E.J. Brouwer and his one-time student Arend Heyting. In the latter's masterwork (Heyting, 1956), we find

> *Logic is not the ground on which I stand. How could it be? It would in turn need a foundation. ... This is the case for every logical theorem: it is but a mathematical theorem of extreme generality; that is to say, logic is a part of mathematics, and can by no means serve as a foundation for it* (Heyting, 1956, p. 6).

## 2. Metatheoretic definitions

Notions constituent to the concept of validity are defined as follows.

**Definitions 2**

1. (*replacing condition*) A *replacing condition* is a formula $\phi(x, X, a)$ in the language of second-order set theory featuring perhaps first-order variables, first-order parameters $a$, as well perhaps as second-order variables and parameters.
2. (*interpretation*) An *interpretation* of a propositional formula is the result of replacing uniformly each of its variables throughout by some replacing condition.
3. (*universal closure*) A *universal closure* of an interpretation $\Phi(x, X, a)$ is the statement that results by prefixing the interpretation with universal quantifiers, restricted to a particular domain or unrestricted, e.g.,

$$\forall x \forall X.\ \Phi(x, X, a)$$

so that all free variables in it are bound and the resulting expression closed.

4. (*generally valid*) A formula of propositional logic is *generally valid* just in case every universal closure of every interpretation of it is true.

Both set validity and class validity, defined as below, are natural restrictions on the notion of general validity. If a formula is generally valid, then it is both set and class valid.

**Definitions 3**

Let *S* be any model of first-order set theory, with, perhaps attached to it, a collection of classes over *S* as well.

1. (*replacing set condition*) A *replacing set condition* is a replacing condition that is first-order, all its variables range over set members of *S*, all its parameters name sets in *S*, and set membership is its sole primitive predicate.
2. (*set interpretation*) A *set interpretation* (over *S*) of a propositional formula is an interpretation of it in which all replacing conditions are set conditions.
3. (*set valid*) A propositional formula is *set valid* whenever, given any model *S*, every universal closure – quantifiers restricted to set members of *S* – of every set interpretation of the formula over *S* obtains.
4. (*replacing class condition*) A replacing class condition is a replacing condition that is first-order or second-order,

all its first-order variables range over $S$, all its first-order parameters name sets in $S$, its second-order variables range over classes over $S$, its second-order parameters names classes over $S$, while set and class membership are its sole primitive predicates.

5. (*class interpretation*) A *class interpretation* (over $S$) of a propositional formula is an interpretation of it in which all replacing conditions are class conditions.
6. (*class valid*) A propositional formula is *class valid* whenever, given any model $S$, every universal closure, its second-order variables restricted to classes over $S$, of every class interpretation over $S$ obtains.

**Proposition 1** Every set replacing condition is a replacing class condition. ■
**Proposition 2** Every set interpretation is a class interpretation. ■
**Proposition 3** Every generally valid formula is class valid. Every class valid formula is set valid. ■

For the sake of the present essay, we adopt classical set-theoretic mathematics – including the general validity of the law of the excluded third – as ambient metatheory. As stated, the final goal is to demonstrate that, even under these generous assumptions, classical set and a recognizable class theory are insufficient to afford conclusive mathematical evidence for the class validity – *a fortiori* general validity – of its own presumptive logical principles. Needless to say, no proprietary intuitionistic theorem, such as the Uniformity Principle (Troelstra and van Dalen, 1988, p. 234) or Brouwer's Continuity Theorem (Troelstra and van Dalen, 1988, p. 307) is here presupposed. These two, among others, each implies at once the invalidity of **TND**.

## 3. A topological model

For decades, it has been common to fancy that all the facts of classical mathematics are encodable as a collection $\mathcal{C}$ of sentences in the language of standard first-order **ZF** set theory, deemed to hold good over the intuitive universe **V** of cumulative sets. We adopt this large assumption – that all of classical mathematics can rightly be rendered in the pared-down language of **ZF** set theory, ready to be captured deductively from extensions of the **ZF** axioms – strictly for the sake of the current exercise. We do not believe it; other people believe it. (One could avoid it by adopting in its stead the much weaker proviso that all classical theorems are representable as a consistent collection of formulae in some single, many-sorted, first-order language.) On that assumption, a classical mathematician's version of "all the mathematics in the world" includes any statements expressible in first-order set-theoretic terms, among them the standard first-order axioms for **ZF** plus large cardinal hypotheses, if desired. Hence, we are sure that "all the mathematics in the world" of $\mathcal{C}$ provides the simple and obvious means for proving that classical formal first-order logic is sound over the relevant universe of sets, and the less simple or obvious means for proving model completeness, i.e., that every classically consistent set of first-order formulae has a Tarskian model.

We also assume throughout that paraconsistent mathematicians will never have their way, and that $\mathcal{C}$ is consistent in classical first-order logic. By model completeness, therefore, there is a model $\mathcal{M}$ of $\mathcal{C}$ with domain $|\mathcal{M}|$. Henceforth, lower case letters $a$, $b$, and the like from the start of the Roman alphabet are parameters for elements of $|\mathcal{M}|$. With the aid of a few more definitions, $\mathcal{M}$ can be enlarged to a universe that includes not only (internal) sets but also classes over $|\mathcal{M}|$, and provides a model of the set theory from $\mathcal{M}$ plus a full, impredicative, second-order class theory. Set and class validity of propositional formulae are then definable with respect to $\mathcal{M}$.

**Definition 4** (*Sierpinski topology and space*) Treated classically, the *topology* $\tau$ that yields *Sierpinski space* on the discrete set $\{0,1\}$ has as its open sets these three:

$$\emptyset, \{1\}, \text{ and } \{0,1\}$$

From here on out, '$\bot$' stands for $\emptyset$ and '$\top$' for $\{0,1\}$. Capital Roman letters from the end of the alphabet such as $U$, $V$, $W$ range over open sets of $\tau$. As is familiar, $\tau$ is closed under the familiar Heyting operations:

1. finite intersection $\bigcap$,
2. arbitrary union $\bigcup$,

3. Heyting implication $V \Rightarrow W$, that is $\bigcup\{U \,\varepsilon\, \tau\colon (U \cap V) \subseteq W\}$, for $V, W \,\varepsilon\, \tau$,
4. Heyting complementation $\sim U$, which maps $U \,\varepsilon\, \tau$ into the $\tau$-interior of the relative complement of $U$ in $\{0,1\}$, and
5. Heyting intersection $\bigwedge$, which takes any collection $U_i$ of $\tau$-open sets into the $\tau$-interior of $\bigcap_i U_i$.

**Definitions 5**

1. ([*topological-valued*] *class*) A (topological-valued) *class* over $\mathcal{M}$ is a function $A$ total over $|\mathcal{M}|$ yielding outputs in $\tau$. The collection of all these (topological-valued) classes over $\mathcal{M}$ is denoted '$|\mathbf{C}(\mathcal{M})|$.'
2. ([*topological-valued*] *set*) For $a$ in $|\mathcal{M}|$, the (topological-valued) *set* $\hat{a}$ over $\mathcal{M}$ is the class over $\mathcal{M}$ such that, for any $b$ in $|\mathcal{M}|$, $\hat{a}(b) = \top$ if $\mathcal{M} \models b \,\varepsilon\, a$, and $= \bot$, otherwise.

Let capital Roman letters from the beginning of the alphabet range over topological-valued classes.

The model $\mathcal{M}$ is assumed to be a model of set theory, hence $|\mathcal{M}|$ is a set itself. The collection $|\mathbf{C}(\mathcal{M})|$ of all topological-valued classes over $\mathcal{M}$ consists of all and only the functions from $|\mathcal{M}|$ into the three-membered set $\tau$. Therefore, it is also a set. Hence, when we speak of 'classes' internal to $\mathbf{C}(\mathcal{M})$ or of 'topological-valued classes,' we are not referring to classes absolutely, but only relative to $\mathcal{M}$. Some of these internal classes, e.g., the denotation of the Russell class abstract, are internally proper – they are not internal sets. In fact, every topological-valued set over $\mathcal{M}$ is also a topological-valued class:

**Proposition 4** Every topological-valued set over $\mathcal{M}$ is also a topological-valued class in $|\mathbf{C}(\mathcal{M})|$.
**Proof:** By definition. ■

Henceforth, when working with $|\mathcal{M}|$ and its extension $|\mathbf{C}(\mathcal{M})|$, we speak simply of sets and classes, respectively, from $|\mathbf{C}(\mathcal{M})|$, less the qualifiers 'topological-valued' or 'internal.'

**Definitions 6** Let $\mathcal{L}$ be a standard second-order language for **ZF** with variables over sets and classes in which

1. first-order variables range over sets from $|\mathbf{C}(\mathcal{M})|$ , while
2. second-order variables range over classes from $|\mathbf{C}(\mathcal{M})|$, and
3. the atomic formulae are of two varieties: $x \varepsilon y$ and $y \varepsilon X$.

Let $\mathcal{L}(\mathcal{M})$ be $\mathcal{L}$ with parameters for sets and classes from $|\mathbf{C}(\mathcal{M})|$.

**Definition 7** (*the topological model* $\mathbf{C}(\mathcal{M})$) *The topological model* $\mathbf{C}(\mathcal{M})$ is the function $\lambda\phi.[\![\phi]\!]$ mapping (closed) sentences of $\mathcal{L}(\mathcal{M})$ into $\tau$ satisfying the familiar recursive clauses introduced in (Tarski, 1938) and elaborated in (Scott, 1968).
for $a$ and $b$ in $|\mathcal{M}|$, $[\![\hat{a} \varepsilon \hat{b}]\!] = \hat{b}(a)$,

for $a$ in $|\mathcal{M}|$ and $A$ a class, $[\![\hat{a}\varepsilon A]\!] = A(a)$,
$[\![(\phi \wedge \psi)]\!] = [\![\phi]\!] \cap [\![\psi]\!]$,
$[\![(\phi \vee \psi)]\!] = [\![\phi]\!] \cup [\![\psi]\!]$,
$[\![(\phi \rightarrow \psi)]\!] = [\![\phi]\!] \Rightarrow [\![\psi]\!]$,
$[\![\neg\phi]\!] = \sim [\![\phi]\!]$,
$[\![\exists x.\ \phi(x)]\!] = \bigcup_{a\varepsilon|\mathcal{M}|}[\![\phi(\hat{a})]\!]$,
$[\![\forall x\ .\ \phi(x)]\!] = \bigwedge_{a\varepsilon|\mathcal{M}|}[\![\phi(\hat{a})]\!]$,
$[\![\exists X.\ \phi(X)]\!] = \bigcup_{\mathrm{A}\varepsilon|\mathrm{C}(\mathcal{M})|}[\![\phi(A)]\!]$, and
$[\![\forall X.\ \phi(X)]\!] = \bigwedge_{\mathrm{A}\varepsilon|\mathrm{C}(\mathcal{M})|}[\![\phi(A)]\!]$.

## 4. Satisfaction, soundness, and class theory

**Definitions 8**

1. (*satisfaction*) The topological model $\mathbf{C}(\mathcal{M})$ *satisfies* a parameterized sentence $\phi$ of $\mathcal{L}(\mathcal{M})$, in symbols $\mathbf{C}(\mathcal{M}) \vDash \phi$, whenever $[\![\phi]\!] = \top$.
2. (*formal derivability*) '$\vdash$' stands for formal derivability over $\mathcal{L}$ or $\mathcal{L}(\mathcal{M})$ specified by the rules of Heyting's formal intuitionistic first-order logic, as in (Troelstra and van Dalen 1988, p. 35ff).

**Proposition 5** (*soundness*) If $\phi \vdash \psi$ and $\mathbf{C}(\mathcal{M}) \vDash \phi$, then $\mathbf{C}(\mathcal{M}) \vDash \psi$.

**Proof:** (sketch) With Gentzen's natural deduction formulation of intuitionistic first-order logic, the proof is straightforward by induction on derivations. ■

Detailed proofs of such foundational results as soundness for Boolean-valued and topological models are available from (Rasiowa and Sikorski, 1963; Bell, 1977; and Grayson, 1979).

**Definition 9** (*abstract class theory*) *Abstract class theory* is axiomatized in $\mathcal{L}$ or $\mathcal{L}(\mathcal{M})$ by the general principle of impredicative comprehension **Com**: for any formula $\phi(\mathrm{y})$ of $\mathcal{L}(\mathcal{M})$ with free variables other than $X$,

$$\exists X.\forall y\,(y\,\varepsilon\,X \leftrightarrow \phi(y)).$$

By adopting **Com**, so formulated, we mean to allow comprehension as well for class-relations of arbitrary arities.

**Proposition 6** $\mathbf{C}(\mathcal{M})\models$ **Com**.

**Proof:** Since $|\mathbf{C}(\mathcal{M})|$ is defined to include all functions from $|\mathcal{M}|$ into $\tau$, it will include the function $A$ mapping each and every $a\varepsilon|\mathcal{M}|$ into $[\![\phi(\hat{a})]\!]$. Therefore,

$$\mathbf{C}(\mathcal{M})\models \forall y\,(y\,\varepsilon\,\mathrm{A} \leftrightarrow \phi(y)).\ \blacksquare$$

Because $|\mathbf{C}(\mathcal{M})|$ satisfies **Com**, the classes of $|\mathbf{C}(\mathcal{M})|$ form a Heyting algebra under $\cap$, $\bigcup$, and relative complementation of classes. The principle **Com**, which is impredicative, is more reminiscent of Kelley-Morse set theory (Monk, 1969) than of Gödel-Bernays. Cf. (Fraenkel and Bar-Hillel, 1958, p. 112).

## 5. Two technical lemmas

The first technical lemma simply eases our way; similar lemmas hold for Boolean-valued models. *Vide* (Bell, 1977).

**Lemma 1** For any sentences $\phi$ and $\psi$ of $\mathcal{L}(\mathcal{M})$,

$$[\![(\phi \rightarrow \psi)]\!] = \top$$

if and only if

$$[\![\phi]\!] \subseteq [\![\psi]\!].$$

**Proof:** Let $x$ be an arbitrary element of the carrier $\{0,1\}$ underlying Sierpinski space. First, assume that

$$[\![(\phi \rightarrow \psi)]\!] = \top$$

and that x $\varepsilon$ $[\![\phi]\!]$. By the definition of $[\![(\phi \rightarrow \psi)]\!]$,

$$x \; \varepsilon \bigcup \{U \; \varepsilon \; \tau : (U \cap [\![\phi]\!]) \subseteq [\![\psi]\!]\}.$$

Hence, there is a $U \; \varepsilon \; \tau$ such that $x \; \varepsilon \; U$ and

$$U \cap [\![\phi]\!] \subseteq [\![\psi]\!].$$

By assumption, x $\varepsilon$ $[\![\phi]\!]$. Therefore, $x \; \varepsilon \; [\![\psi]\!]$. Since $x$ was arbitrary, it holds generally that

$$[\![\phi]\!] \subseteq [\![\psi]\!].$$

Conversely, assume that $[\![\phi]\!] \subseteq [\![\psi]\!]$. It follows that, for any $U \varepsilon \tau$,

$$U \cap [\![\phi]\!] \subseteq [\![\psi]\!].$$

Therefore, by definition of the $\Rightarrow$ operation,

$$([\![\phi]\!] \Rightarrow [\![\psi]\!]) = [\![(\phi \rightarrow \psi)]\!] = \top. \blacksquare$$

Now in view is the principal lemma governing the semantic relation between the original model $\mathcal{M}$ of first-order set theory – satisfying "all the mathematics in the world" – and the topological model $\mathbf{C}(\mathcal{M})$ for set theory with classes under full comprehension **Com**. Such a relation is reminiscent of the fact that, over Scott's topological model for analysis, elementary arithmetic is absolute (Scott, 1968).

**Lemma 2** For each strictly first-order sentence $\phi(\hat{a})$ of $\mathcal{L}(\mathcal{M})$ with all parameters $\hat{a}$ referring to sets of $|\mathbf{C}(\mathcal{M})|$,

$$[\![\phi(\hat{a})]\!] = \begin{cases} \top & \text{if } \mathcal{M} \vDash \phi(a) \\ \bot & \text{otherwise.} \end{cases}$$

**Proof:** (By induction on $\phi(\hat{a})$ using classical metamathematics)

(i) For $\phi$ atomic, by definition.

(ii) If $\mathcal{M} \vDash (\phi \wedge \psi)$, then $\mathcal{M} \vDash \phi$ and $\mathcal{M} \vDash \psi$. By the inductive hypothesis,

$$[\![\phi]\!] = \top = [\![\psi]\!].$$

Therefore,

$$[\![\phi \wedge \psi]\!] = \top.$$

On the other hand, assume that $\mathcal{M} \nvDash (\phi \wedge \psi)$. *Without loss of generality*, we can assume that $\mathcal{M} \vDash \phi$. Then, $\mathcal{M} \nvDash \psi$ and, by induction, $[\![\psi]\!] = \bot$ as well as

$$[\![(\phi \wedge \psi)]\!] = \bot.$$

(iii) If $\mathcal{M} \vDash (\phi \vee \psi)$, then $\mathcal{M} \vDash \phi$ or $\mathcal{M} \vDash \psi$. By the inductive hypothesis, if $\mathcal{M} \vDash \phi$, then

$$[\![\phi]\!] = [\![(\phi \vee \psi)]\!] = \top.$$

Likewise for the subcase $\mathcal{M} \vDash \psi$.

On the other hand, if $\mathcal{M} \nvDash (\phi \vee \psi)$, then both $\mathcal{M} \nvDash \psi$ and $\mathcal{M} \nvDash \psi$. By induction,

$$[\![\phi]\!] = \bot = [\![\psi]\!],$$

and $[\![(\phi \vee \psi)]\!] = \bot$.

(iv) Assume that $\mathcal{M} \vDash (\phi \rightarrow \psi)$. If $[\![\phi]\!] = \top$, then the inductive hypothesis gives $\mathcal{M} \vDash \phi$. In that case, $\mathcal{M} \vDash \psi$ and $[\![\psi]\!] = \top$. Hence, $[\![(\phi \rightarrow \psi)]\!] = \top$.

On the other hand, if $[\![\phi]\!] = \bot$, then

$$[\![\phi]\!] \subseteq [\![\psi]\!]$$

and, by **Lemma 1**,

$$[\![(\phi \rightarrow \psi)]\!] = \top.$$

Classical reasoning then yields the result.

For the converse, assume that $\mathcal{M} \not\models (\phi \rightarrow \psi)$. So, $\mathcal{M} \models \phi$ while $\mathcal{M} \not\models \psi$. By induction, we know that $[\![\phi]\!] = \top$ and $[\![\psi]\!] = \bot$. Therefore, $[\![(\phi \rightarrow \psi)]\!] = \bot$, appealing once more to **Lemma 1**.

(v) From $\mathcal{M} \models \neg\phi$ it follows that $\mathcal{M} \not\models \phi$ and, hence, that $[\![\phi]\!] = \bot$ and

$$[\![\neg\phi]\!] = \top.$$

From $\mathcal{M} \not\models \neg\phi$ it follows classically that $\mathcal{M} \models \phi$. Therefore, $[\![\phi]\!] = \top$ and, by definition of $\lambda\phi.[\![\phi]\!]$ and the inductive hypothesis,

$$[\![\neg\phi]\!] = \bot.$$

(v) $\mathcal{M} \models \exists x.\phi(x)$ just in case, for some $a \;\varepsilon\; |\mathcal{M}|$ , $\mathcal{M} \models \phi(a)$. It follows from the inductive hypothesis that there is an $\hat{a} \;\varepsilon\; |\mathbf{C}(\mathcal{M})|$ such that

$$[\![\phi(\hat{a})]\!] = \top.$$

Therefore,

$$\bigcup_{a\varepsilon|\mathcal{M}|} [\![\phi(\hat{a})]\!] = \top,$$

and

$$[\![\exists x.\phi(x)\ ]\!] = \top.$$

In addition, $\mathcal{M} \not\models \exists x.\phi(x)$ just in case, for all $a\ \varepsilon\ |\mathcal{M}|$, $\mathcal{M} \not\models \phi(a)$. It follows that, for all $a\ \varepsilon\ |\mathcal{M}|$,

$$[\![\phi(\hat{a})\ ]\!] = \bot\ .$$

Therefore, $[\![\exists x.\phi(x)\ ]\!] = \bot$.

(vii) Lastly, $\mathcal{M} \models \forall x.\phi(x)$ just in case, for all $a\ \varepsilon|\mathcal{M}|$, $\mathcal{M} \models \phi(a)$. Hence,

$$\forall a\ \varepsilon\ |\mathcal{M}|\ [\![\phi(\hat{a})]\!] = \top.$$

So, $\bigcap a\varepsilon|\mathcal{M}|\ [\![\phi(\hat{a})\ ]\!] = \top$. Therefore, by definition of $\lambda\phi.[\![\phi]\!]$,

$$[\![\ \forall x.\phi(x)\ ]\!] = \top.$$

On the other hand, if $\mathcal{M} \not\models \forall x.\ \phi(x)$, then (classically speaking) there is an a $\varepsilon\ |\mathcal{M}|$ such that $\mathcal{M} \not\models\ \phi(a)$. It follows by the inductive hypothesis that,

$$\exists a\varepsilon|\mathcal{M}| \text{ such that } [\![\phi(\hat{a})]\!] = \bot.$$

Therefore,

$$\bigcap_{a\varepsilon|\mathcal{M}|} [\![ \phi(\hat{a}) ]\!] = \perp,$$

and

$$[\![ \forall x\, \phi(x) ]\!] = \perp. \blacksquare$$

## 6. Two principal theorems

**Theorem 1** The entire mathematics of $\mathcal{C}$ encoded in the language of set theory and holding in $\mathcal{M}$ obtains as well in $\mathbf{C}(\mathcal{M})$.
**Proof:** Immediate from **Lemma 2**. ■

Whatever claims belong in $\mathcal{C}$ – the axioms of familiar **ZF** set theory most likely, together with the Axiom of Choice or Determinacy (Mycielski and Steinhaus, 1962) or whatever other set-theoretic principles are desired and consistent with them – will hold for the sets in $\mathbf{C}(\mathcal{M})$. As assumed previously, the mathematics internal to $\mathcal{M}$ contains demonstrative means sufficient to prove the soundness of classical propositional logic, alternatively, to prove the validity of **TND** in a more direct fashion. Therefore, we see that
**Corollary 1** $\mathbf{C}(\mathcal{M})\models$ **TND** is set-valid. In other words, "all the (classical) mathematics in the world" is consistent with and certifies that **TND** is set-valid. ■

It remains only to note that the classical first-order and (intuitionistic) second-order, class-theoretic mathematics internal to $\mathbf{C}(\mathcal{M})$ do not suffice to prove the class validity, and hence the general validity, of the law of the excluded third.

**Theorem 2** $\mathbf{C}(\mathcal{M}) \not\models \forall X.\forall x(x \,\varepsilon\, X \vee \neg\, x \,\varepsilon\, X)$.
**Proof:** Let $A$ be the class in $|\mathbf{C}(\mathcal{M})|$ such that, for any $a \varepsilon |\mathcal{M}|$, $A(a)=\{1\}$. It is clear that

$$[\![\forall x(x \,\varepsilon\, A \vee \neg\, x \,\varepsilon\, A)]\!] = \{1\}.$$

Indeed, for any class $B$ in $|\mathbf{C}(\mathcal{M})|$,

$$\{1\} \subseteq [\![\forall x(\, x \,\varepsilon\, B \vee \neg x \,\varepsilon\, B)]\!],$$

since for any such $B$ and any $\hat{a}$,

$$\{1\} \subseteq [\![(\hat{a} \,\varepsilon\, B \vee \neg \hat{a} \,\varepsilon\, B)]\!].$$

So,

$$[\![\forall X \forall x(x \,\varepsilon\, X \vee \neg x \,\varepsilon\, X)]\!] = \{1\}.$$

Therefore,

$$\mathbf{C}(\mathcal{M}) \not\models \forall X.\forall x(x \,\varepsilon\, X \vee \neg x \,\varepsilon\, X). \blacksquare$$

**Corollary 2** $\mathbf{C}(\mathcal{M}) \not\models$ **TND** is generally valid. ■

**Corollary 3** "All the mathematics in the world" cannot prove that **TND** is generally valid.

**Proof:** Classical set-theoretic mathematics (the collection of first-order statements in $\mathcal{C}$), which includes the set validity of **TND**, as well as a full impredicative class theory satisfying **Comp** are consistent with the failure of the general validity of **TND**. ■

Full separation for sets by classes, as in Kelley-Morse theory (Monk, 1969),

$$\forall X \forall y\, \exists z \forall x (x\ \varepsilon\ z \leftrightarrow (x\ \varepsilon\ X \wedge x\ \varepsilon\ y)),$$

does not hold within $\mathbf{C}(\mathcal{M})$. Of course, because truth in $\mathbf{C}(\mathcal{M})$ agrees with that in $\mathcal{M}$ for all first-order conditions, separation holds for all predicative or first-order abstractors in $\mathcal{L}(\mathcal{M})$. Such restricted versions of separation or Bernays's Axiom of Subclasses (Fraenkel and Bar-Hillel, 1958, p. 114) will already be familiar to researchers in constructive set and type theory. Cf. (Aczel, 1978). Similar remarks apply to the Replacement Scheme or Bernays's Axiom of Substitution (Fraenkel and Bar-Hillel, 1958, p. 114).

## 7. Going one better

Let $\sigma$ be the topology determined by the $\leq$-upward closed sets of natural numbers under their canonical ordering $\leq$. Define from structure $\mathcal{M}$ the model $\mathbf{C}(\mathcal{M})$ as above, but with $\sigma$ in place of the Sierpinski space topology $\tau$. Proofs of the preceding propositions, lemmas, and theorems all go through, so that $\mathbf{C}(\mathcal{M})$ satisfies the classical mathematics presupposed at the outset, together with the set validity of **TND**, all holding in $\mathcal{M}$. This time, the law of the excluded third not only fails of general validity on classes in $\mathbf{C}(\mathcal{M})$, but the negation of the class validity statement obtains.

**Theorem 3** $\mathbf{C}(\mathcal{M}) \models \neg\forall X \forall x\, (x \;\varepsilon\; X \vee \neg x \;\varepsilon\; X)$ .
**Proof:** Let $\mathbb{N}$ be the set of natural numbers. Check that, with $\sigma$ replacing $\tau$ in the construction of $\mathbf{C}(\mathcal{M})$,

$$[\![\forall X \forall x (x \;\varepsilon\; X \vee \neg x \;\varepsilon\; X)]\!] = \bot.$$

To see this, note that, for the upward-closed, open set $n\uparrow = \{m \;\varepsilon\; \mathbb{N} : \mathrm{n} \leq \mathrm{m}\}$, and class $A$ such that, for all a $\varepsilon$ $|\mathcal{M}|$, $A(a) = n\uparrow$,

$$[\![(\hat{a} \;\varepsilon\; A \vee \neg \hat{a} \;\varepsilon\; A)]\!] = n\uparrow. \blacksquare$$

**Corollary 4** "All the mathematics in the world" plus an impredicative intuitionistic class theory is consistent with $\neg\forall X \forall x\, (x \;\varepsilon\; X \vee \neg x \;\varepsilon\; X)$, hence, with the statement that the general validity of **TND** is false outright. $\blacksquare$

A die-hard classical mathematician cannot escape the force of these arguments by stomping his or her foot and insisting that classical mathematics must be encoded in language of a second-order (Shapiro, 1991), rather than first-order, set theory, that is, in a theory of sets and classes such as Kelly-Morse or Gödel-Bernays. It is now clear that even the second-order classical mathematician cannot prove the law of the excluded third to be generally valid, since it remains consistent with all the second-order mathematics in the world, if it is consistent, to assume that the law of the excluded third fails over hyperclasses, in other words, a further collection of third-order classes that are themselves collections with members that are the original, possibly 'Henkinized' (Henkin, 1950) classes and sets.

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