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BERNHARD RIEMANN: A FORERUNNER OF CLASSICAL ELECTRODYNAMICS (An Historical Epistemological Approach)

In the mid-19th century, progress in both mathematical and experimental physics indicated a synthesis of Classical Electrodynamics was imminent. The general framework of physical thinking developed to a point at which that landmark in the history of Natural Science in the 19th century became possible. By 1867, the synthesis had been accomplished, owing to the work of James Clerk Maxwell (1831—79) and, independently, to Ludwig Lorentz of Copenhagen (1829—91). The two gifted physicists had proceeded along very different roads. While Maxwell followed Michael Faraday's idea that electric and magnetic fields are primary entities of the physical reality under investigation to give this doctrine its perfect from, Lorentz continued along the line indicated by German mathematicians of Göttingen—Karl Friedrich Gauss (1777—1855), Wilhelm Eduard Weber (1804—91), but notably Bernhard Riemann (1826—66)—who put the concept of potential at the basis of a construction which turned out to be of mechanical inspiration. A mechanical model of electromagnetic phenomena employs of course some very useful concepts as variational calculus, Lagrange function, gradient, Laplacean, divergence, curl, scalar potential, vectorial potential, force, mechanical work, energy, momentum, etc. English physicists did not dispensed completely of mechanical representations of electromagnetic phenomena, for at that time Newtonian Mechanics wielded an all-inclusive sovereignty, firmly relying as it did on the Kantian apriorism of space and time and actio-in-distans doctrine of interaction. Whereas physicists on the Continent turned generally to mechanics of point-like bodies, trying to build-up a general electromagnetic theory based on a fundamental two-body interaction, the English tended to focus attention on the mechanics of continuous media, succeeding in carrying out their programme and reaching a satisfactory explanation which imposed aether (a concept borrowed from a Frenchman—Augustin Fresnel) as a physical reality—until the advent of Einsteinian relativity disproved that concept, at least in its classical formulation. It is in such hydrodynamic considerations, for instance, that William Thomson (Lord Kelvin, 1824—1907) as early as in 1846 established a set of differential equations, now known as the Maxwell-Lorentz equations, which define the relationship between the magnetic induction vector $\vec{B}$ and the vectorial potential $\vec{A}$.

$$\vec{B} = \nabla \times \vec{A}, (\vec{B} = \mu \vec{H}). \tag{1}$$

Maxwell called the vector $\vec{A}$ "electronic intensity." We have to point out that the two starting-points for the subsequent development of electromagnetic theory (proceeding either from twopoint-like-body interaction or from disturbances appearing in a certain fluid) did not exist exclusively in Britain and in the Continent, respectively. In fact, the "English" approach was applied by some physicists in Germany and France, and we shall see later that Bernhard Riemann and Ludwig Lorentz took advantage of both standpoints to pick valuable elements from both. The framework of research in Electrodynamics had become quite large by the middle of the 19th century, so we confine ourselves only to some major findings to make it easier to understand the work of Riemann—the main subject of this study. The equation of electrostatic potential

$$\Delta \phi = -4\pi \varrho / \varepsilon \tag{2}$$

was well known. It had been derived long time before, in 1813, by Siméon Denis Poisson (1781—1840), the outstanding French mathematician and mechanician of the Restoration Epoch. In 1843, Mich... Faraday—one of the most brilliant Anglo-Saxon minds—gave the first convincing experimental proof about the conservation of electrical charge. In modern terms, this result may be written as

$$\frac{\partial \varrho}{\partial t} + \nabla \vec{j} = 0. \tag{3}$$

(Incidentally, electrical current density $\vec{j}$, still considered a source of the vectorial potential, appeared in the explicit form

$$\vec{j} = \varrho \vec{v} \tag{4}$$

for the first time only in 1883 in Fitzgerald's report to a British Association meeting in Southport. This assertion, made by Edmund Whittaker [1], should be accepted with some reserve. Further investigations should be undertaken, say of Hermann von Helmholtz's papers who seems to have been the first physicist envisaging the concept of electron.) In a 1856 communcation to the Cambridge Philosophical Society, J. C. Maxwell presented Faraday's law of electromagnetic induction in a "local form" which is valid even now

$$\nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0. \tag{5}$$

The same physical phenomenon of electromagnetic induction, was studied mathematically in 1857 by Gustav Kirchhoff of Heidelberg (1824—1887), who followed quite a different path, adopting the German School's structuralist point of view which gave prominence primarily to the concept of potential,

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}. \tag{6}$$

Kirchhoff's equation (6) leads to Maxwell's equation (5) if the Curl-operator is applied to (6) and is accounted for in virtue of Thomson's equation (1). Another remarkable German physicist and mathematician, Franz Neumann of Königsberg (1798—1895), arrived at equation (1) independently of Thomson in 1848 at the same time establishing the equation of the vectorial potential of a steady distribution of electrical currents

$$\Delta A = -4\pi\mu\vec{j}/c. \tag{7}$$

Formula (7) for the magnetism of steady currents plays a similar role to that of formula (2) for electrostatics. Since at such electrical currents the divergence of current density vanishes, that the divergence of the potential vector itself should vanish accordingly. So, accounting for the mathematical identity

$$\text{Laplacean (of a vector)} = \text{Grad}\cdot\text{Div} - \text{Curl}\cdot\text{Curl} \tag{8}$$

and also for equation (1), we can transform equation (7) to obtain

$$\nabla\times\vec{H} = 4\pi\vec{j}/c. \tag{9}$$

Equation (9), together with equation (5) were published by Maxwell in 1868. A mathematical consequence of equation (1) is

$$\nabla\vec{B} = 0. \tag{10}$$

Today we would say this equation expresses the absence, in Nature, of free magnetic charges (i.e., charges existing outside magnetic dipoles). In Riemann's 1854 physical mathematical papers we come across the correct form of continuity equation (3) but without its specification (4). Concerning the mathematical apparatus of the epoch, the operators Curl and Div. were frequently used by George Stokes of Cambridge (1819—1903), especially in his 1849 *Dynamical Theory of Diffraction.* On the other hand, major contributions to the potential theory were made by Franz Neumann and Bernhard Riemann himself. To complete the picture of theoretical electromagnetism Riemann's times, we must mention the remarkable works about magnetic interaction between electrical steady currents undertaken by Ampère (1823) and Wilhelm Weber (1846), analysed by Riemann in his lectures on *Schwere, Elektrizität und Magnetismus* held at the University of Göttingen in 1858—63. His *Vorlesungen* (published posthumously by Karl Hettendorf in 1867) impress readers by their mathematical discipline and their quality in general, which were in no way inferior to the greatest theories of the time. Faraday's works, which seem to be debatable to Riemann, and those of Neumann were equally well known to Riemann. Although we have no peremptory evidence of that at present, Riemann may well have been aware of the valuable works of Hermann von Helmholtz of Berlin (1821—94), perhaps the most brilliant mind among German physicists of the epoch, and of Rudolf Julius Emanuel Clausius of Bonn (1822—88), concerning interaction between electrical charges in motion.

Now, let us try to assess cautiously Riemann's own role in his times, which eventually led to a synthesis of Classical Electrodynamics. Most textbooks of Electrodynamics today usually call the set of equations of the relevant branch of Physics as "Maxwell-Lorentz equations", which is likely to deepen the confusion between the two famous Lorentzes—the Danish Ludwig and the Dutch Hendrik. For convenience these equations are often referred to simply as "Maxwell's equations". Moreover, in the typical modern axiomatic manner of presenting electrodynamics other contributions, apart from those of Maxwell and Hendrik Lorentz, are sometimes left unacknowledged. So, students often get a confused picture of contributions made by some great physicists such as Ampère, Laplace, Gauss, Weber, Clausius, Volta, or even Faraday, L. Lorentz and Helmholtz. These circumstances are rendering our job of redefining Riemann's place and role in the history of Electrodynamics even more difficult. Of course, we have no intention of overestimating anyone in any epoch or country. All we want to do is to establish, along with the historical facts relating to a great discovery, all contemporary contributions to a problem, which were often left unfinished, not necessarily due to some built-in contradiction but to some fortuitous event either in the life of the person concerned (as was the case of Riemann) or in the process of accomodation to and acceptance of other unconventional ideas by the scientific community of the epoch (as was the case of Nilakanta's mathematical number *e* in mediaeval India).

The outcome of such a competition of contribution and the future of a new and revolutionary idea greatly depend on its discoverer's ability to put the idea in a logically convincing form, on his perseverance in promoting it and bringing it home to his contemporary scientific community, and also on the necessary links of the new element to the corpus of science in which it naturally belongs and to which it introduces the relevant information to meet what can be called "scientific demand." The difference between an inventor or discoverer and a forerunner is perhaps that the latter leaves his work in an incomplete version, indeed even in the fact that the forerunner, despite having completed his work, has little confidence in his own discovery and does not work hard enough to promote it in his own time. In such cases, the strong authority of commonly accepted ideas of the epoch, of the paradigms on which the science of the epoch relies, prevents the accomplishment of a revolutionary act.

In Riemann's case, it was not his character or his intellectual scope which eventually prevented him from enjoying the glory which naturally comes in recognition of outstanding creative acts in Science—it was unfortunately his bad health. A merciless disease, phthisis, struck him at the age of 37. Three years later, in 1866, when Maxwell and Ludwig Lorentz were widely acknowledged as great men of science, Riemann, still in the prime of life died of consumption. This happened in the same year in which he was elected F. R. S. His not-too-extensive body of work is essential to establish his place among the world's great mathematicians. Suffice it to mention just some common terms named after him: (maximally homogeneous) Riemannian space (with positive curvature), dzeta function of Riemann (in the theory of prime numbers), Riemann's curvature tensor (of fourth order), Riemann's conformal transformation (of the complex plane), etc. Riemann's electrodynamic theoretical contributions were largely as valuable guides for his successors, Ludwig Lorentz being the first to take advantage of it. But his scientific methodology and his basically heuristic approach are as valuable as ever and now, a century later, they are strikingly modern. His physicist's intuition was so profound that he often used it to simplify intricate problems of pure mathematics, without arriving—as Felix Klein remarked—at wrong conclusions.

A study of the activity and mode of thinking of a forerunner of a great scientific discovery may be interesting from several points of view. It is not only necessary to "rehabilitate" Riemann as a requirement of objectivity in describing any aspect of general history, but also, and above all, to acknowledge the great benefit for the philosophy of science, for the epistemology of the creative act in science. To illuminate unexplored facets of a great discovery, to penetrate intimately the nature of the thought, to scrutinize all circumstances objective and psychological, the scientist's attitude towards his own work, and, lastly, to carry to the end all unfinished approaches to establish the causes of a partial (or total) failure—all these in the light of modern research strategy, are undertakings of inestimable worth.

Such critical penetrations of the history of science, based upon reconstructions and historical and epistemological models, may be of great value for those responsible for scientific policy design in various respects (selecting candidates for research work, picking the most efficient of several possible ways towards a given objective, adopting a more tolerant attitude towards unconventional ideas). Such models may lead us to more coherent versions of some theories or demonstrations of obvious usefulness in instruction. Our investigation in mathematical analysis in mediaeval India is a case in point. The derivation of the number e we proposed in virtue of a historical and epistemological model composed of elements which existed (or virtually existed) in the epoch, has a definite advantage over the more common derivation procedure based on the limit $_{n\to\infty}(1+\frac{1}{n})^n$. While this limit is an expression of a social demand historically concretized in a "continuous interest," our derivation naturally stems from a necessity to develop mathematical analysis. Apart from the demand, the quest for the limit appears as no more than a fortuitous inspiration and the student may be in some doubt about the logic behind the progress of knowledge. We hope that this brief outline of reasons conclusively show that our interest in Riemann's work as a landmark in the history of Science is useful and may help the Present to learn from the Past for a better forecasting of the Future.

Bernhard Riemann was born in 1826 in Germany. At the age of just 25, he received his doctor's degree at Göttingen University. His dissertation is now considered to be a fundamental contribution to differential geometry, general relativity and cosmology. In 1854, he wrote two studies on the subject of electrodynamics: *Über die Gesetze der Vertheilung von Spannungselectrizität* and *Neue Theorie des Rückstandes in Electrischen Bindungsapparaten.* The local form of the electrical charge conservation equation is used in both papers. Also in 1854, Riemann wrote his Habilitationsschrift, and was already regarded as one of the most promising young mathematicians. His masters included Gauss, Wilhelm Eduard Weber (1804—91) and Peter Gustav Lejeune Dirichlet (1805—59). The English philosopher Herbert Spencer (1820—1903) greatly influenced Riemann's philosophical position on evolutionism, unity of the universe and rationalist comprehension. A decisive turning-point in Riemann's scientific career came in 1859, when he succeeded Dirichlet at the Mathematical Department of Göttingen University and begun his famous lectures on Gravitation Electricity and Magnetism (1859—63). By that time, he had demonstrated his remarkable talents as theoretical physicist and especially in electrodynamics; in 1858, he put forward, in a rigorous mathematical form, the idea of electric interaction propagating at the speed of light in empty space. That Riemann's approach can be seen not only from his adequate quantitative expression for the retarded action, but also from his intuitive choice of potential, and not force, as the primary physical quantity propagating at light speed. In Newtonian physics, of course, potentials are of subsidiary use as entities describing forces acting upon a certain body. This reversed order of priorities is a basic feature of Riemann's approach, which makes it entirely different from the alternative Maxwellian approach. His valuable contribution to Electrodynamics, regarding the equation of electrical potential

$$\Delta\phi - \frac{\varepsilon\mu}{c^2}\frac{\partial^2\phi}{\partial t^2} = -4\pi\varrho/\varepsilon \tag{11}$$

was included in *Ein Beitrag zur Electrodynamik* presented to the Academy of Göttingen, a paper Riemann first withdrew but eventually published, still in 1858, in "Poggendorf Annalen der Physik und Chemie" (vol. 131). Equation (11) embraces equation (2) of electrostatic potential as a particular case. We think it an elementary duty towards objectivity and a tribute to Riemann to call equation (11) "Riemann's equation of electrical potential."

It is, perhaps, opportune now to make a digression on the high reputation Riemann enjoyed among German scientists long after his death, even in connection with his research in theoretical physics. In his opening address in Vienna on 27th September 1894, at a Meeting of the Society for Natural Sciences, Felix Klein (1849—1925) said:

Je dois vous prévenir d'abord que Riemann s'est beaucoup occupé, et d'une manière très suivie, de considérations physiques. Elévé dans la grande tradition dont les noms réunis de Gauss et Wilhelm Weber sont le symbole, influencé, d'autre part, par la philosophie de Herbert Spencer, il a toujour et à maintes reprises, travaillé à recherche d'une forme mathématique sous laquelle pourraient être exprimées, d'une manière unique, les lois auxquelles tous les phénomènes naturels sont soumis.

To come back to Riemann's outstanding discovery formulated in equation (11), Riemann did not confine himself to the mathematical aspect, i.e. a formal generalization of equation (2). To include dynamic aspects in it, he embarked on an attempt to identify the very mechanism of the interaction propagation. The question Riemann presumably confronted was this: since potential is just a mathematical abstraction just what is really propagating at the speed of light? To answer this difficult question, he found himself compelled to introduce McCoulagh's idea of aether and to assume that interactional information propagates, together with the transverse wave of aether disturbance, at the speed of light. After along investigations, in 1861 he concluded that aether density is proportional to $\sqrt{\varepsilon\mu\phi}$, while aether current is proportional to $c/\sqrt{\varepsilon\mu}\cdot\vec{A}$. Writing a continuity equation to account for the conservation of universal aether, Riemann could then formulate another fundamental equation of classical Electrodynamics

$$\nabla\vec{A}+\frac{\varepsilon\mu}{c}\frac{\partial\phi}{\partial t}=0. \tag{12}$$

Among physicists, this equation is known as "Lorentz's gauge condition." Indeed, Ludwig Lorentz obtained this equation in 1867 as part of a set of equations equivalent to those of Maxwell. Riemann's priority is acknowledged by Edmund Whittaker in his *History of the Theories of Aether and Electricity* (1953, vol. I, p. 291). Wittaker also points out in that book that Lorentz's own approach was in line with Riemann's reasoning:

The procedure which Lorentz followed was that which Riemann had suggested in 1858—namely, to modify the accepted formulae of electrodynamics by introducing terms which, though too small to be appreciable in ordinary laboratory experiments, would be capable of accounting for the propagation of electrical effects through space with a finite velocity (*op. cit.*, p. 268).

Two important points must be made: first, by discovering equation (12), Riemann was on the brink of synthesizing all classical Electrodynamics, which makes him virtually one of its founders; second, the method Whittaker outlines attributing it to Riemann is as valid now as ever, and many modern research areas, to mention but invariantive mechanics, indirectly benefit from it. The essential step forward towards a synthesis of Electrodynamics, in the Lorentzian version, was to obtain an equation of the vectorial potential homologous to equation (11)

$$\Delta\vec{A}-\frac{\varepsilon\mu}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-4\pi\mu\vec{j}/c. \tag{13}$$

Riemann did not accomplish this, but he was much closer to this decisive point than appears at a first sight. Indeed, by applying the D'Alembert operator to equation (12) $\left[\Box \equiv \Delta - \frac{\varepsilon\mu}{c^2}\frac{\partial^2}{\partial t^2}\right]$, by reversing thereafter the order of operations and going back to equation (11), one obtains an equation comparable with the equation of charge conservation (3). Thus, as a direct consequence of eqs. (12), (11) and (3), and without adding other new concepts or conjectures, anybody with Riemann's mathematical background, and acute intuition and power of organizing of his facts was in a position to arrive at the intermediary equation

$$\nabla\left\{\Delta\vec{A} - \frac{\varepsilon\mu\partial^2\vec{A}}{c^2\,\partial t^2} + 4\pi\vec{j}/c\right\} = 0. \tag{14}$$

At this stage of demonstration, a comparison between eqs. (14) and (7), of the same vectorial quantity but referring to the particular case of steady currents, gives the necessary psychological argument for accepting eventually equation (13) as appropriate for the vectorial potential, covering all the possible cases. In this way, the propagation at light speed of the vectorial potential is ensured as a product of the adopted procedure—an aspect of nature confirming the legitimacy of the approach. But Ludwig Lorentz, who probably did not know equation (12) (derived by Riemann in his 1861 lectures but not published before his death), followed a different road to arrive at equation (13), which was still framed in the Riemannian paradigm (to perform small amendments upon equations of mathematical physics in such a way as to promote some basic ideas systematically without coming into conflict with the corpus of empirical data of the time). In 1867, Lorentz wrote down solutions to eqs. (2) and (7) as three-dimensional integral representations, and ascertained that "retardation" of the sources was a necessary mathematical amendment to be made in view of Riemann's idea of propagation of electrical effects at light speed. In other words, the quantities ϱ and $\vec{j}$ are to be taken in the integral not at the instant t, but at the instant $t_{\mathrm{Ret}} = t - \sqrt{\varepsilon\mu}/cR$, where R is the distance between an arbitrary point in space the coordinates of which are integration coordinates, and another arbitrary point in which we want to calculate the values of potentials. Thus, the speed of the afore-mentioned propagation turns out to be $V = c/\sqrt{\varepsilon\mu}$. The correct expression of t_{Ret} was obtained, before Lorentz, by Riemann, in the case of electrical potential. The derivation of the field equations when the potential equations and the connections between fields and potentials are known is merely a question of some relatively easy mathematical transformations. For instance, equation (10) is a trivial consequence of equation (1). Equation (5) follows from equation (6) after applying the curl operator, if, in addition, equation (1) is taken into account. The derive the equation of electrical induction (Gauss, 1845?)

$$\nabla\vec{D} = 4\pi\varrho, \qquad \vec{D} = \varepsilon\vec{E} \tag{15}$$

use equation (11) to which adequate terms are added and subtracted so that the vanishing quantity, expressed in (12), should be formed. Subsequent inspection of the definition of electrical field in terms of potentials (equation 6) leads to the expected result. Finally, to obtain equation (9) in a form holding for all cases take equation (13), consider operational identity (8), add and subtract adequate terms so that the vanishing quantity (12) should be formed, and eventually express $\vec{B}$ and $\vec{E}$ in terms of potentials [eqs. (1) and (6)]. The resulting equation

$$\nabla \times \vec{H} - \frac{1}{c}\frac{\partial \vec{D}}{\partial t} = 4\pi \vec{j}/c \tag{16}$$

was first derived by Maxwell in 1868 and published in "Philosophical Transactions" (vol. 158, p. 643). That result brought glory on Maxwell, whose prestige rose still more when he discovered that the set of equations (5)+(10)+(15)+(16) is selfconsistent in the sense foreseen by Riemann, namely in propagation conjecture, opposed to the Newton's tacit assumption of "actio in distans," which was fully verified when Heinrich Rudolf Hertz (1857—94) demonstrated experimentally the existence of electromagnetic waves using an electrical oscillator devised and built by himself. While this great discovery of 19th-century science is currently being associated with Maxwell and L. Lorentz Riemann's role as a forerunner should not be overlooked. It should be pointed out that L. Lorentz derived, in 1867, only the electrodynamic formulas for charges and currents in empty space, when $\varepsilon = \mu = 1$. A generalization of this for $\varepsilon \neq 1$, $\mu \neq 1$ was a later step owed to many contributors (Maxwell among them). As soon as the complete set of field equations was written down the synthesis of Electrodynamics was accomplished. That, however, was a partial synthesis, because a complete classical picture of electrodynamic phenomena necessarily must include the behavior of point-like electrical charges in certain electric and magnetic fields. This second task proved, during the second half of the 19th century, even more difficult to perform than the first one. A complete synthesis of Electrodynamics was performed only in 1904, when H. Lorentz derived the force formula keeping his name

$$\frac{d}{dt}\left(\frac{m_0 \vec{v}}{\sqrt{1-\frac{\vec{v}^2}{c^2}}}\right) = q\left(\vec{E} + \frac{\vec{v}}{c} \times \vec{H}\right). \tag{17}$$

It refers to a point-like body endowed with a rest mass m_0 and an electrical charge q. At the beginning of the 20th century, not only H. Lorentz, but also Joseph Larmor and Henry Poincaré arrived at a complete synthesis of Electrodynamics, anticipating in this respect the Relativity theory. But our job is to establish Riemann's contribution to the problem of motion of charged point-like bodies. In particular let us look at his thinking and methodology in that subject. In principle, formula (17) covers all possible cases of motion of point-like electrical charges. It may be derived by starting with a Lorentz invariant action principle (as Max Planck did in 1906), whereby the controversy over the actio-reactio principle in electromagnetic phenomena is explained away. However, due to the interaction propagation at a finite speed, potentials which enter expressions of fields depend on what are quite intricate functionals; in fact it is almost impossible to write down accurate and finite expressions for the motion equations. It was only in 1963 that Alfred Schild succeeded in deriving the Lagrange function of a relativistic rotator, made up of two point-like bodies with arbitrary masses and charges, and related to the frame of inertia center, as an analytical expression in terms of direct Euclidean invariants of motion. He used Maxwell's equations of electrodynamics and, after long and tedious calculations, managed to sum up the infinite series coming from the retardation effect. The mentioned invariants are all independent Euclidean invariants made up of position $(\vec{r}_1, \vec{r}_2)$ and velocity $(\vec{v}_1, \vec{v}_2)$ vectors of the two point-like bodies, preserving this quality under three-dimensional space rotation, under the change of the origin of inertia frame by a constant vector, and under space reflection. There are altogether six of them:

$$\alpha_1 \equiv \tfrac{1}{2}\vec{v}_1^2,\ \alpha_2 \equiv \tfrac{1}{2}\vec{v}_2^2,\ \varrho \equiv \vec{v}_1 \cdot \vec{v}_2,\ \eta_1 \equiv \vec{v}_1 \vec{r},\ \eta_2 \equiv \vec{v}_2 \cdot \vec{r},\ r \equiv |\vec{r}|.$$

To come back to Riemann's epoch, the motion problem was entered upon at that time both via the force concept and the Lagrange interaction function, the second way presenting the obvious advantage of ensuring equality between action and reaction and the existence of the first ten integrals of motion. Riemann adopted a Lagrange function inspired by that of Weber, in which he replaced relative radial velocity by relative velocity. By this change you can obtain the same magnetic energy between two closed loops wandered by steady electric currents as those in Weber's original theory. The Laplacean axiom on such a loop and a magnetic dipole being equivalent was used by Weber, Riemann and Helmholtz, with a view to going over from an elementary interaction act between two charges in motion to the interaction of two circuits of current. Helmholtz was the first who adopted a linear combination between the functions due to Weber and Riemann, as a Lagrange function appropriate for a mechanical system of two point-like charges in motion.

$$\begin{aligned} L_{\text{Helmholtz}} &= \frac{K_1}{K_1+K_2} L_{\text{Weber}} + \frac{K_2}{K_1+K_2} L_{\text{Riemann}} = \\ &= (m_{0_1}\alpha_{0_1} + m_{0_2}\alpha_2) - \frac{q_1 q_2}{r} + \frac{1}{C_E^2}\frac{q_1 q_2}{r} \times \\ &\times \left[\frac{K_1}{K_1+K_2}\frac{\eta_1\eta_2}{r^2} + \frac{K_2}{K_1+K_2}\zeta\right] + \frac{1}{C_E^2} f. \end{aligned} \tag{18}$$

The first term in the above formula is the classical kinetic energy, the second term is the static Coulombian energy, the third term (containing arbitrary constants K_1, K_2) is the magnetic energy, while the last term is a corrective function following from theoretical considerations. By adequately choosing the function f we can ensure the inertial motion of the mass center of the mechanical system made of the two point-like bodies. There is no way around this requirement, because suppressing quantity f will result in a small acceleration of the mass center. However, the structure of this quantity can be determined only after a definite option concerning theoretical mechanics. If, for instance, we accept the Newtonian Mechanics as a true and accurate theory of Nature, then at infinite interparticle separation the Lagrange function must reduce (apart from a constant quantity, irrelevant at this stage of analysis) to the partners' kinetic energy. Therefore, f must depend on or in such a way as for giving $f(r = \infty) = 0$, but, if must also vanish when one of the two charges goes towards zero. Finally, the imposition of inertial motion of mass center completely determines the function

$$f_{\text{classical}} = -\frac{1}{2}\frac{q_1 q_2}{r}\left[\frac{K_1}{K_1+K_2}\left(\frac{\eta_1^2}{r^2}+\frac{\eta_2^2}{r^2}\right)+\frac{K_2}{K_1+K_2}2(\alpha_1+\alpha_2)\right. \tag{19}$$

The motion is thus described by some Euler & Lagrange variational equations coming from a Lagrange function $L = L(\alpha_1, \alpha_2, \zeta, \eta_1, \eta_2, r)$. The structure of this function is chosen so as to ensure the invariance of motion equations under Galileo transformations. The constant C_E, called "electrodynamic constant" is, in this approach, not a universal but a specific constant assigned to electromagnetic interaction. At least formally, the possibility arises to reconcile the "actio-in, distans" character of Newtonian Mechanics with the finite speed of electromagnetic interaction as described by Riemann. But this consistency, accomplished within the framework of classical mechanics, soon turned out to be illusive. Indeed, if expression (19) is inserted in the Lagrange function for the noble reasons of saving the classical mechanics, then an additional eletrodynamic interaction is included which, however, does not vanish when one of the two velocities $(\vec{v}_1, \vec{v}_2)$ goes to zero.

Helmholtz, the theorist who carried up this research after Reimann's death along the road mapped out by Weber and Riemann, was the first to resolve—aware as he was of the contradiction—to suppress the quantity f. That was an open declaration of war on classical mechanics and the possibility for a new deep change in the science of mechanics was emerging. Was Helmholtz fully aware of his position? Was he realizing the necessity of changing the basic statements of theoretical mechanics? These questions are difficult to answer. At all events, further investigations are necessary if we are to conclude finally whether the contradiction of classical mechanics was explicit at those times or remained a latent logical contradiction to this day.

But, for our epistemological purposes, even a logical contradiction suffices to lay a bridge between early investigations concerning the motion of charged point-like bodies and modern research on theoretical mechanics initiated by Octav Onicescu in Bucharest and developing the "invariantive mechanics" (working for $v \ll c$, $v \approx c$). Without being aware of Weber's approach, or of the investigations of Reimann and Helmholtz, Octav Onicescu developed an entirely new theoretical Mechanics. In this form it was later possible to derive a correct formula for a two-body rotator before Schild by Nicholas Ionescu-Pallas and Liviu Sofonea which extended the theory of Onicescu – working out both a Lagrangean version and a Hamiltonian one – and finally, reaching accurate results equivalent to a rigorous formulation of relativistic analytical mechanics. The role of the limiting velocity is played in this approach by the electrodynamic constant C_E, and Romanian physicist Dragomir Hurmuzescu's experiment establish accurately equality between the electrodynamic constant and light speed in empty space acquires fundamental importance.

We can safely assume that Weber, Riemann and Helmholtz made "invariantive mechanics" *avant la lettre*. Coming back to formula (18), we have to suppress the quantity f of potential (interactional) origin and insert instead another quantity f, this time of kinematic origin. The form of this new corrective function may be foreshadowed by combining the kinematic character of that quantity (which precludes from its structure any invariant connecting the two partners of the system, and, at the same time, prescribes to it an essential additiveness) with arguments from dimensional analysis. This, brings us to the formula

$$f_{\text{Non-Classical}} = K_3\left[m_{0_1}\alpha_1^2 + m_{0_2}\alpha_2^2\right] \tag{20}$$

The inertia principle may thus be saved (i.e., the acceleration of the mass center may be avoided) provided that we give up the Newtonian concept of mass as a certain constant quantity and accept that body's mass may depend on velocity and on its interaction with other bodies. In other words, kinetic and potential energies of a body contribute to its inertia. By inserting expression (20) in (18), and by asking to comply with the inertia principle and to ensure the existence of all the first ten integrals of motion, we conclude that such a non-classical solution does exist, provided that the three dimensionless constants are subject to some restrictions, namely: $K_1 = K_2$, $K_3 = 1/2$. Unlike the classical case, this time the avoidance of mass center acceleration proportional to C_E^{-2} does not remove accelerations in higher orders of approximations. Accordingly, the procedure must be taken over for determining corrective terms proportional to C_E^{-4}, C_E^{-6}, etc. Such terms can no longer be built up without interactional contributions. Summing up all kinetic terms from various orders of approximation is no special problem, but summing up all potential terms turns out to be an extremely complicated job. This is the reason why no such attempt, using the Maxwell equations and the Lorentz transformations, has been a success yet. Unlike this, the alternative procedure, relying on invariantive mechanics (the historical roots of which are to be found in the epoch of Weber, Riemann, Helmholtz, Clausius and F. Neumann) and on its synthetic approach (dispensing with intricate functionals resulting from retardation), can well produce remarkable results.

An accurate and finite Lagrange function of two point-like bodies with arbitrary rest masses and charges referred to the inertia centers

$$L = -m_{c_1}c^2\left(1-2\frac{\alpha_1}{c^2}\right)^{1/2} - m_{0_2}c^2\left(1-2\frac{\alpha_2}{c^2}\right)^{1/2} - \frac{q_1 q_2}{r}\frac{\left(1-\frac{\zeta}{c^2}\right)}{\left(1-\frac{\zeta}{c^2}+\frac{\eta_1\eta_2}{c^2 r^2}\right)^{1/2}}. \tag{21}$$

Other physical and epistemological results are obtained using the same methodology. A long way had to be gone from Riemann's 1858 study about the finiteness of interaction propagation to equation (21) obtained by Nicholas Ionescu-Pallas in 1977. Within this itinerary of thinking, which in the "historiography" was not completely persecuted, but which can be conceived and achieved in the epistemological–history (meta history) there are two decisive instants marked by two outstanding minds deserving full appreciation, namely Bernhard Riemann as a forerunner of classical electrodynamics, and Octav Onicescu, a founder of Invariantive Mechanics. Their lifetimes were surprisingly similar and are linked with a scientific problem of outstanding significance.

REFERENCES

[1] E. Whittaker — "A History of the Theories of Aether and Electricity," Volume I — The Classical Theories, London, Thomas Nelson Ed., 1951.

[2] B. Riemann: *Oeuvres Mathematiques*, Paris, Gauthier—Villars, 1898 (French edition of the second German edition of Riemann's *Mathematical Works*, Leipzig, 1876).

[3] B. Riemann: *Schwere, Elektrizität und Magnetismus*. Nach den Vorlesungen von ... bearbeitet von Karl Hattendorf, Hannover, Karl Rümpler Verlag, 1876.

[4] L. Lorentz: *Oversigt over det K. Fanske Vid. Selskaps Forhandlinger*, 1867, p. 26; "Ann. d. Phys.," CXXXI, 1867, p. 243; "Phil. Mag.," XXXIV, 1867, p. 287.

[5] H. Arzelies: *La dynamique relativiste*, Paris, Gauthier—Villars, 1957 (The Riemann's contributions to Electrodynamics are pointed out).

[6] J. C. Maxwell: *Treatise on Electricity and Magnetism*, 1873.

[7] N. Ionescu-Pallas: *Elements of the Theory of Electricity, Magnetism and Light*, Institute for Atomic Physics, Bucharest, 1974 (in Romanian).

[8] M. Planck: *Das Prinzip der Relativität und die Grundgleichungen der Mechanik*, "Deutsch. Phys. Ges.," 1906, p. 136—141; *Zur Dynamik bewegter systeme*, "Sitz. Berichte Akad. Wiss.," 1907, p. 542—570; "Ann. Physik," Bd. 26, no 8, 1908, p. 1—34.

[9] L. Sofonea: *Les models historiques et epistemologiques et la recherche de l'évolution des sciences exactes*, "ΕΛΕΥΟΕΡΙΑ," 1979, no 2, p. 500—523 (Athens, Grecce).

[10] L. Sofonea and N. Ionescu-Pallass: *New Commentions Between Classical Mechanics and Electrodynamics*, "Buletinul Universitatii din Braşov," Vol. XXII, Seria C – Matematica, Fizica, Chimie, 1980, p. 217—226.

[11] L. Sofonea and N. Ionescu-Pallas: *Logic of the Historical Development of Science and Inner Logic of Physical Theories*, Central Institute of Physics, Bucharest 1981, 40 pag. (Work devoted to the XVI-th International Congress of the History of Science, Bucharest, 26th August—3rd September, 1981).

[12] O. Onicescu: Invariantive Mechanics, Springer Verlag, Vienna, 1978. See also: *Mecanica invariantiva și cosmologia*, Edit. Acad. R. S. R., București 1976.

[13] N. Ionescu-Pallas: *Invariantive Approach to Analytical Mechanics of Point-like Bodies*, "Revue Roumaine de Physique," Tome 23, nr. 9, 1978, pp. 995—977. See also: *Relativitate generala si Cosmologie*, Edit. Stiintifica și Enciclopedica, București 1980.

[14] N. Ionescu-Pallas, L. Sofonea: *The Manysidedness of the Scientific Approach, Historical-Epistemological Modelling, Examples from Modern Physics*, Noesis, V. Edit. Acad. R. S. Romania, București 1979.

[15] N. Ionescu-Pallas, L. Sofonea: *The Genisis of the Mathematical Entities, A Historical-Epistemological Modelling Example*, Noesis, III. Edit. Acad. R.S. Romania, București 1975.

[16] I. Sofonea, N. Ionescu-Pallas: *From the History of High Speed Mechanics. A Historico-Epistemological Modelling of the Relations Between Classical Electrodynamics and Invariative Mechanics*, Noesis, VIII, București 1984. *New Connections Between Classical Mechanics and Electrodynamics*, Annales St. Univ. Al. I. Cuza Jassy, sect. I. b. Phys. T. XXXII 1985.

[17] L. Sofonea, N. Ionescu-Pallas: *The Efficiency of Historical-Epistemological Models. Examples from Modern Physics*, [in:] *The IX-th International Congress of History of Science*, Edinburgh 1971. *The Possibility of Conceptualizing the Transcendent Number e in the Mediaeval Indian Mathematics*, [in:] The IX-th International Congress of History of Science, Edinburgh 1971.

[18] L. Sofonea: *Contributions to the History of the Concept of Infinity* [in Romanian]. Doctoral Thesis in History of Science, University of București, 1986.