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THE „ITALIAN ALGEBRA" IN LATIN AND HOW IT SPREAD TO CENTRAL EUROPE: GIOVANNI BIANCHINI'S *DE ALGEBRA* (ca. 1440)*

Summary

Giovanni Bianchini's *Algebra* is one of very few 15th-century treatises on this subject written in Latin, and the only one incorporated in an astronomical work. It is the „second treatise" of the *Flores Almagesti*, a summa of astronomy divided into eight, nine or ten „treatises" in keeping with different manuscript traditions. In the *Flores* the *Algebra* follows a treatise on arithmetic and precedes one on proportions. The first three treatises taken together are a mathematical introduction to Bianchini's presentation of astronomy.

Bianchini (ca. 1400–ca. 1470), by his own account, came to learn algebra very early in life, and he did so for two reasons. First, he had his job to take care of, secondly, he had an enthusiasm for mathematical astronomy. Almost all his life was spent in business, till 1427 as merchant in Venice, later in Ferrara as administrator of the estate of the marques d'Este. In this latter job, to which he devoted forty years or so of his life, Bianchini divided his time between the discharge of his duties at the court and astronomy. He compiled astronomical tables and tables of trigonometric functions, wrote the *Canones tabularum*, and spent time making astronomical observations using instruments of his own design. All that is known from different extant documents, including his correspondence with Regiomontanus in 1463–1464.

Bianchini spent fifteen years writing the *Flores*, between 1440 and 1455. Later he went on rounding up his work, probably till 1460, putting together some details and rewriting some chapters. Bianchini's habit of working on different astronomical treatises at the same time (e.g., on the *Canones tabularum eclypsium* along with the *Flores Almagesti*) makes it difficult to establish the exact chronology of his writings. Many questions remain unanswered to date.

The dating of the mathematical part of the *Flores*, however, particularly the *Algebra*, is less complicated. Bianchini had every reason to put it down to paper at the beginning of his work on the *Flores*, that is, early in the 1440s, and it was probably for a first time he explained in writing the mathematical knowledge he acquired some twenty years earlier, in the Venetian milieu of the *scuole d'abbaco*.

The reception of the *Flores* seems to have begun at about the time Bianchini and Regiomontanus started their correspondence. In a letter of January 12, 1463, Bianchini tells Regiomontanus of a copy of the *Flores* circulating in Venice. By the beginning of 1464, Regiomontanus writes Bianchini that he is still in possession of the *Flores*. The copy he mentioned, with his annotations, has been kept at the Library of Cracow University from the end of the 15th century. Also nearly

the end of the 15th century fragments of the *Flores* appeared in Cracow (including a complete text of the *Algebra* though). They were annotated by Martin Biem of Olkusz, professor at the time Copernicus studied at Cracow University (ms. BJ 601).

The seven extant copies of the Algebra, the Cracow ones included, were produced in Italy between ca. 1460 and ca. 1490.

Introduction

In the 15th century algebra was first of all of interest to merchants and bankers rather than learned men at universities. It used to be taught at what were called *scuole d'abbaco*, where, unlike at universities, the vernacular was used rather than Latin. Those leaving the abacus schools were specialists in financial matters connected with property division, crafts cooperative and merchant's businesses, stock-taking, or bank interest.

Sometimes an abacus school would be headed by an outstanding mathematician, and then other individuals interested in developing algebra would rally to him. Such was the case of Masolo da Perugia (fl. 2nd half of the 14th cent. and of Benedetto da Firenze (fl. 1460). They tried to solve equations higher than square ones, and indeed they were successful when solving cubic equations of the type $ax^3+bx^2+cx=d$ for a, b, c, $d \in N$, and $a=1$, when $b^2/3a=c/a$.[1] They also studied equations of the type $ax^{2n}+bx^n+c=0$. At the schools they tried their hand at irrational numbers, submitting them to arithmetical operations. The abacists thus certainly helped pave the way to the theory of real numbers that appeared several centuries later.

The dynamics of teaching mathematics for the needs of the emerging modern economics led up to a paradox as time went by: it was not university science that was being popularized but popular abacus schools stirred the development of theoretical algebra leaving their mark on it. That, however, occurred only in the 16th century. Abacus handbooks of earlier origin extant in 15th-century university collections are very rare indeed. The Jagellonian Library, for example, has in its collection of manuscripts used by Cracow students at the turn of the seventies and eighties of the 15th century one brief treatise, *Arithmetica mercatorum*. It is a textbook of „merchant's arithmetic" in Latin (BJ 2729, f. 76r–77v), probably a translation from Tuscan or some other vernacular.

Even artists, like professional merchants, outdid universities in getting involved with 15th-century abacus schools. Piero della Francesca, the painter and theoretician of perspective, also wrote an algebra textbook called *Trattato d'abaco*.[2] It was presumably from abacus schools that men like Filippo Brunelleschi, Paolo Toscanelli or Leon Battista Alberti drew their mathematical knowledge. Leonardo da Vinci, as a disciple of Luca Pacioli's, have drawn his mathematics from the same source.

Early in the 15th century Giovanni Bianchini found that algebra could be used in astronomical calculations, and Johannes Regiomontanus noticed that some thirty years later. Bianchini had got his knowledge of algebra at one of the abacus schools of Venice, where he spent his youth as a merchant dealer.[3]

The best schools, however, worked in Florence and in southern France. The Florentine tradition is seen in hundreds of manuscripts written in the Tuscan vernacular that are now kept mainly at the Biblioteca Medicea Laurenziana in Florence, but also in the Biblioteca degli Intronati of Siena, and at the Vatican Library. A repertory of those codices published by Warren van Egmond[4] and studies by a Siena University team of historians of mathematics can give readers an idea how enormous that historically unique effort to teach „applied mathematics" was.[5] That supplies clinching evidence confirming views of Scipione dal Ferro, Cardan, Ferrero or Raffaele Bombelli who all acknowledged 15th-century Italian abacists' great role in the development of algebra.[6]

Another powerful current of „mercantile mathematics" developed in southern France. While it had its roots in Italy, those schools could boast sometimes original accomplishments. Evidence of that is found in some recent editions of treatises written in the langue d'oc,[7] but above all in Nicolas Chuquet's of Lyon *Triparty en la science des nombres* (1484) which was discovered only in the 19th century.[8]

While the scientific production of Renaissance algebraists had little in common with universities, the origins of European algebra in the Latin civilization several centuries before were linked to universities. Algebra was known at Western European universities as early as in the mid-12th century.[9] On the other hand, it is a fact that algebra was ignored in university *curricula*. All through till the 16th century algebra has not been found in lists of subjects taught at universities, (with the exception of the university in Leipzig?) and there is a striking shortage of algebraic treatises written in Latin, preserved in universities collections of manuscripts. The above-mentioned „Cracow" *Arithmetica mercatorum* is an exception there.

The presentation of algebra by Giovanni Bianchini, designed as a mathematical introduction to an astronomical work, is a unique landmark in algebraic literature. Bianchini's *Algebra* in the second book of the *Flores*, is preceded by arithmetic and along with *the Arithmetic* introduce readers to the science of proportions and elements of plane and spherical trigonometry.[10]

The *Flores Almagesti*, containing the most comprehensive discussion of mathematics before Luca Pacioli's *Summa* (1494), is a work of a man who spent his life managing the estate of the noble family in Ferrara, levying taxes and engaging in diplomatic activities.[11] Though Bianchini was active outside the university milieu, and yet all he ever had written was destined for it.

1. The significance of the Cracow collection for the discovery of the *Flores Almagesti*

The *Flores Almagesti*, a work long believed lost,[12] was discovered by Ludwik Antoni Birkenmajer more than eighty years back in the manuscript collection of Cracow University Library (Biblioteka Jagiellońska, ms. BJ 558). In a note published in German, Birkenmajer described the manuscript and its content along with several brief passages from it.[13] His comments on Bianchini's *Algebra* showed it was quite an important discovery.[14] Birkenmajer's communication was noticed by historians of science, some of whom, including Antonio Favaro, had long befriended and cooperated with the Polish author. How significant his little study had become to the history of science can be seen from the fact that it keeps being mentioned in any major publication concerning 15th-century mathematical or astronomical literature, even though since Birkenmajer's discovery five more copies of the *Flores* have been found across Europe.[15] But that case also shows that important discoveries sometimes go unheeded, because they fail to get worked through or incorporated in every new synthetic histories of the particular disciplines of science: for more than eighty years now the brief and necessarily sketchy presentation of the algebraic part of the *Flores* has never been expanded or supplemented with an edition of the text.[16]

Apart from the Cracow manuscript, Bianchini's *Flores* are today known to exist in five copies, kept in France, Italy and the Vatican. All copies were produced in Italy and in each of them the *Algebra* is preceded by a textbook on arithmetic. As for the *Algebra*, there is a seventh copy, which was produced separately from the whole work. Cracow has two copies, one in the above-mentioned first-discovered manuscript of the *Flores*, signature BJ 558, and the one described as the "seventh copy", in manuscript BJ 601.[17] The copy of *Algebra* in ms. BJ 601 seems to be also of Italian provenience.

The *Algebra* of ms. BJ 558 is most probably the copy of which it is known from Bianchini's correspondence with Regiomontanus that the latter owned it from February 1464,[18] or at least that Regiomontanus could do what he liked with it, if he freely put down notes about his perusal of Bianchini's work on the margin. Those notes, two of which were quoted by Birkenmajer, were identified to be in Regiomontanus' own hand by E. Zinner.[19] Nothing is known about the roads on which the manuscript had arrived in Cracow, yet there is nothing surprising about the presence of that manuscript in the library of Cracow university, if you recall Regiomontanus' friendship and cooperation in mathematics and astronomy with Cracow astronomer Marcin Bylica of Olkusz, or Marcin Bylica's well-known concern about expanding the library of his *alma mater*. The Jagellonian library in Cracow keeps to this day all works by Regiomontanus Bylica used to donate to it in the order in which they were written.[20] Furthermore, Bylica's own library along with a collection of astronomical instruments was bequeathed in his will to the university.[21] The existence in the ms BJ 558 of Regiomontanus' glosses have so far been ignored in Polish studies of the matter[22] A comparison of some of marginal notes from the BJ 558 with Regiomontanus' Latin autographs of the 1460s, has led me to accept Zinner's attribution as correct.[23]

As for the next copy of the *Algebra* kept in Cracow (the one transcribed separately from the whole work yet along with several other mathematical fragments from it) existing in ms. BJ 601[24] it is linked with the names of two Cracow lecturers, Martin Biem of Olkusz and Mikołaj Młodszy (The Younger) of Wieliczka nicknamed Mleczko. In this codex Martin gave the title *Arismetrica algebre* to the originally untitled copy of the work of Bianchini.[25] As for Nicholas Mleczko, apart from making notes in margins, he put his signature on f. 181v.[26] Inspection of the codex, including the identification of water-marks, shows it was produced after 1474. So, unlike the codex BJ 558 which is dated for the turn of the 1450s and 1460s, codex BJ 601 was produced after Bianchini's death.

2. The content of Bianchini's *Algebra*

a. The terms

Bianchini's *Algebra* sets off in a classic manner, meaning here it goes back to Latin translations of al-Khwarizmi dating of the 12th century, as well as to Leonardo Fibonacci's work of the 13th century. Namely, just as in those other works the presentation starts with the introduction of terms *res, census, cubus, census de censu* – which mean an unknown in the first, second, third and fourth power, respectively. Bianchini himself uses descriptive language introducing no symbols. That shows another of his dependences on the medieval tradition. But, as it will be seen later, some of the 15th century readers of the *Algebra* introduced symbols in marginal notes.

Bianchini's approach fits in the tradition of a geometrical algebra going back to Euclide, in which demonstration of arithmetic operations is done using segments and planes.[27] Thus Bianchini defined the term *res* as corresponding to a straight line but also as corresponding to a root (a side of a square); *census* a square of a number, as a square plane, and *cubus* as a cube. Since no physical entity was available to denote the fourth degree, Bianchini simply put forward the proposition: "*census de censu* is a square raised to the second power". And, he concludes: "everything originates from the root, that is from *res*".[28]

In the history of the European algebra it was very rare indeed before the 16th century that anyone treated algebra arithmetically, an approach that was common in Moslem mathematics. Exception is Diophantus, in the 3rd century, Jordanus of Nemore in 13th, and Jean de Murs in the 14th century. In the 15th century, recent studies seem to show, there was a move back to Euclide and the geometric tradition in algebra. This is probably even more true of Regiomontanus than of Bianchini, even though the former of the two enthused about the work of Diophantus, the latest rediscovery of Greek science at the time.[29]

b. Basic operations on algebraic expressions

As mentioned before, Bianchini preceded his discussion of algebra with a presentation of arithmetic, the first book of the *Flores Almagesti*, with an exposition of the four basic operations along with rising to powers and extracting square and cube roots. In the same presentation Bianchini discussed rules of operation with negative numbers (speaking, for example, of „subtractive addition"),[30] treated arithmetically surds, or irrational numbers and showed how to calculate their approximate values.

After those preparatory remarks a systematic discussion of algebra could begin with definitions of operations on algebraic expressions. They were simple operations on polynomials. In division, incidentally, Bianchini pointed out algebraic expressions could occur in the denominator and repeated his earlier rule for fraction-by-fraction division by multiplying the dividend by the reciprocal of the divisor. Here are examples, in notation used today, which Bianchini adduced to explain operations with polynomials:

$(4+2x)(9+3x)$; $(5+3x)(8-2x^2)$; $(6-3x)(4-8x)$;
$(4+5x+2x^2)(7+5x^2)$; $(8+7x):3/2x$; $8x^2:4x$.

Bianchini listed the order in which the operations are to be carried out and indicated ways of reducing expressions. In discussing multiplication he used the common rule of „cross" multiplication which he marked in the margin:

4 plus 2res

9 plus 3res

In the BJ 601, the copyist explaining this rule used symbols in the margin: a zero crossed with a vertical dash above which the copyist wrote *caracter numeri*, and the word *res* replaced by the letter r ending with a loop. In the ms. Perugia 1004 little squares are used as symbols.

c. Algebraic rules for simple equations (simplicia)

It was al-Khwarizmi who first distinguished six canonic algebraic equations, three simple and three compound ones. The simple equations,

(I) $ax^2=bx$; (II) $ax^2=c$; (III) $bx=c$,

are called, in Bianchini's Latin description, *census aequantur radicibus; census aequantur numero*; and *radices aequantur numero*, respectively.

All arithmetic problems solved algebraically were first to be reduced to one of the canonic expressions where $a=1$; $b, c > 0$. The canonic form was obtained by transformations including the rules: al-dzhabr, for moving expressions to the other side of the equation with the opposite sign, and al-muquabala, for reducing similar expressions.

Bianchini appends to his algebra textbook ten exercises with solutions of simple and compound equations.[31] The solutions are meant to be illustrations of a procedure demonstrated geometrically in a chapter called *Regulae conclusionum ad practicam algebrae in simplicibus*. Each simple equation is a variant within any of three classes distinguished by al-Khwarizmi.[32]

d. Algebraic rules for compound equations (*composita*)

This portion of Bianchini's treatise, called *Regulae conclusionum in compositis cum demonstrationibus in superficie plana*, is the main body of the presentation. As announced in the title, Bianchini adduces rules of solutions along with their geometric proofs. The subtitle promises to deal with basic rules (*regulae fundamentales*) unlike rules called „adherent" (*adhaerentes*) to the basic rules to be discussed further on in the section. Canonic forms of square equations described as *composita* are, according to al-Khwarizmi,

(IV) $ax^2+bx=c$; (V) $ax^2+c=bx$; (VI) $bx+c=ax^2$.

Bianchini in his presentation follows this order showing numerical examples:

(IV) $x^2+16x=36$; (V) $x^2+24=14x$; (VI) $8x+20=x^2$.

The point of that was to illustrate and prove the rules geometrically rather than to find numerical values of the roots, even though numerical solutions are given for each equation.

Equations of classes (IV) and (VI) always have one and only one positive root (Bianchini never envisages the eventuality of negative root even though, as mentioned before, he used negative numbers in arithmetic operations).[33]

For class (V) of equations Bianchini named all three possibilities: no solution, a solution with only one root, and with two positive roots.[34]

The above-mentioned *regulae adhaerentes* are an attempt, an unsuccessful one yet repeatedly cropping up in abacists' studies, to expand rules of solving square equations to cubic equations.[35] Moreover, Bianchini solved there equations of higher degree reducible to square equations, $ax^3+bx^2=cx$, and $ax^4+bx^3=cx^2$.

e. Practical applications of algebra: exercises

Of the seven copies of the Algebra, two, including the one preserved in ms. BJ 601, are terminated with sets of exercises for the *simplicia* and the *composita* (cf. note 31). Exercises for the application of compound equations concern mainly problems originating from the *scuole d'abbaco* (except for the search of numbers that are in definite mutual proportion to one another). Bianchini demonstrated the solutions to each equation. For example, exercise No. 2 (symbolic notation is mine).

„A merchant purchased a commodity for a given price. Later he sold the commodity at a profit of 20 ducati. He invested the [initial] capital together with that 20 ducati in a different good, which he also sold at a profit at the same proportion as the profit made from the first investment. He found he had a total of 125 ducati. I am to find the value of the first investment."

[Solution:] Suppose the investment is res [x]. Using that investment he made a profit of 20 ducati in the first operation, so he had x plus 20 ducati, which he reinvested to make a profit of the same proportion [as in the first case].

If for the first time from the x he got x+20, then these are to be multiplied by each other: (x+20) (x+20), with the product being $x^2+40x+400$, which is to be divided by x, and that must be equal to 125.

So I multiply, in keeping with the second [rule] of this [chapter], 125 by x to obtain 125x equal to $x^2+40x+400$. I subtract 40x from both sides of the equation, and get,

$$x^2+400=85x.$$

According to the second rule of this [chapter, i.e., concerning equation V], I am halving [the coefficient of] x, obtaining a half equal to 42 1/2, which when squared yields a product equal to 1806 1/4, the root of which is 37 1/2.

I can supply two answers, for I can say: x is equal to [this] root 37 1/2 subtracted from one half of the coefficient of x which is 42 1/2. That was the original investment, that is, 5 ducati. [Also, the other possibility] the root [37 1/2] added to one half etc. [37 1/2 + 42 1/2] will be that first investment, that is, [I am going to obtain] 80 ducati.

The same will be found in the final conclusion, that is, the result will be 125 ducati between profit and investment.

And this is precisely what I have demonstrated in this connection."

3. Are manuscripts of Bianchini's *Algebra* proof of universities teaching algebra in the 15th century?

The question of algebra as a subject of university teaching has yet to be answered. Latest studies took into account Central European universities: Leipzig, Vienna, Erfurt.[36] Earlier Ludwik Antoni Birkenmajer asked the same question in connection with Cracow University as he mentioned mathematical manuscripts appearing there at the end of the 15th century in relation to a bequest to Martin Bylica of Olkusz.[37] His question, now a hundred years old, still belongs in a broader context of conjectures about the teaching of mathematics in 15th-century universities.

The extant copies of Bianchini's *Algebra,* all coming from Italy, were prepared in the three decades between ca. 1460 and 1490.[38] Not all were linked to universities though. It seems that, apart from the two Cracow codices, the Bologna codex signature 19(292) and the Perugia one signature 1004 may have had links to university milieus. No evidence points to any university connection of the Vatican manuscript Vat Lat 2228, the most beautiful of all, completed with illuminations and meticulously transcribed, probably first of all to cater to somebody's bibliophilic taste. The copyist put his signature at the end: *Johannes Carpensis civis Ferrarie*, with the date 4 December 1470. Nor is there any trace of the use of the text for teaching purposes in the Paris *Algebra*, Bibliothèque Nationale signature ms. lat 10253, transcribed in Naples in 1481 and 1487 by bibliophile and editor Arnold of Brussels who was on the search for works in Italy, mainly scientific ones, to add them to his private collection or for publication. In the same line belongs the Vatican *Algebra* from Queen Christine's of Sweden collection, signature Vat. Reg. Lat. 1904, which again bears no trace of glosses or provenience notes that might be interpreted as evidence of having been used in teaching. This last-named codex, moreover, while written in distinguished Latin, has numerous omissions which in many cases make it the meaning of the presentation unintelligible if left to go without a corrigendum.

Conclusion

Manuscripts BJ 558 and BJ 601 showed that towards the end of the 15th century, and at the turn of the centuries at the latest, the Cracow University milieu had at least two transcripts of Bianchini's *Algebra* for its use.

Already in the early 1450s Cracow scholars began to copy Bianchini's works, you can say as soon as they were produced. Bianchini's Planetary tables, for example, the version of 1452, were copied by Jan Zmora of Leśnica in Perugia the following year.[39] This went on through to the end of the century.[40] It would not be surprising at all, then, should earlier copies of the *Flores* than those dispersed in Italy and in France be found to have been preserved to this day in Cracow. Moreover, Martin Bylica of Olkusz is unlikely not to have known the *Flores* already in the 1460s. In 1462 he read astronomy at Bologna University, which had contacts to Ferrara. At the same time, he cooperated with Regiomontanus in the calculation of astronomical tables known as *Tabulae directionum* which largely depended on a similar set of astronomical tables Bianchini calculated some twenty years before. Nor can it be ruled out that the Jagellonian Library copy of the *Flores* with Regiomontanus' annotations actually belonged to Marcin Bylica already at the time he worked with students in Bologna.

The Bianchini treatise presented here, in what sense is it related to teaching *curricula* in Cracow, or to mathematical skills of Cracow scholars in the 15th century? There is no definite answer to these queries yet. The first thing to do is to scrutinize the lectures in mathematics by Cracow professors, preserved in manuscripts now in libraries in Cracow, Vienna, Leipzig, Milan, Oxford, Paris etc., to determine to what extent they dealt with substance that went beyond 13th- or 14th-century *algorismi*. The only studies of this kind we have got to date concern mathematical treatises written by Marcin Król (Rex) of Żurawica (mid-1440s).[41] The mathematics in his astronomical work, *Summa super tabulas* has yet to be studied though.[42] Other mathematical treatises by late 15th-century astronomers to take a close look at are above all those by Wojciech (Albertus) of Brudzewo and of Jan of Głogów, lecturer in mathematics at universities in Cracow and in Vienna. Furthermore, several mathematical treatises, fragments and notes I have come across in Cracow collection of manuscripts have ever been mentioned in published studies.[43]

The *Flores Almagesti* took some twenty years to write, but the parts devoted to arithmetic, algebra and trigonometry must have been ready by the year 1440, because Bianchini made references to their contents in the first version of his *Canones Tabularum primi mobilis* preserved in Florence (Biblioteca Medicea Laurenziana, ms. Ashb. 216). The *Canones* are possibly even older than the planetary tables dedicated to Leonello d'Este in 1442. Indeed, Bianchini must have been versed in algebra even before 1427, the year Leonello entrusted him with the management of the finances of his court.

However, what has been established to date makes it difficult to determine from which decade of the 15th century Cracow scholars had become intimate to the mathematics of the *Flores*.

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Giovanni Bianchini's *Algebra* precedes the trattati d'abaco of Benedetto da Firenze (1463) and of Piero della Francesca (1476), both of which, contrary to Bianchini's work, were written in vernacular. It also precedes by more than three decades Luca Pacioli's treatise on algebra included in his *Summa de aritmethica, geometria, proportioni e proportionalita* (1494). To assume that Bianchini's work, or the work of that kind, was the source of both, Piero della Francesca's and Pacioli's treatises in algebra, seems a plausible approach. This possibility seems to have been ignored in discussions to date about the dependence of Pacioli's *Summa* on Piero della Francesca's *Trattato d'abaco*.

The circumstance that astronomical problems are absent from the algebraic part of the *Flores* makes Bianchini's *Algebra* more similar to the standard *libri d'abaco*. That resemblance is even more striking in the case of two copies of the *Algebra* (preserved in the ms BJ 601 and BN Lat 10253) that close, like all Italian *libri d'abaco*, with a set of problems of commercial arithmetic. As did authors of some of the *trattati d'abaco*, Bianchini supplied geometrical proofs of the algebraic rules he presented. His knowledge of algebra as showing in the *Flores* perhaps did not place Bianchini much above the level of competence 15th-century teachers at the *scuole d'abbaco* could boast, yet it was certainly better than that most of 15th-century university professors had.

Notes

* The paper was prepared on the basis of a critical edition of Bianchini's *Algebra* to appear in „Studia Copernicana".

I wish to thank The Harvard University Center for the Italian Renaissance Studies Villa I Tatti, Florence, as I was able to collect sources dealing with Bianchini's scientific activity during my one-year stay there. At a later stage of my work, the Centro Studi e Incontri Europei in Rome enabled me to verify a first typescript of Bianchini's *Algebra* with codices kept in Rome and Bologna, which makes me grateful to Ms. Wanda Gawrońska. Lastly, I wish to thank the Centro di Cultura Italiana in Warsaw for enabling me a one month stay at University of Perugia and the Sidney M. Edelstein Center for the History and Philosophy of Science, Technology and Medicine at Hebrew University in Jerusalem where I could use the excellent library to collect the secondary literature pertinent to the subject.

I prestented the content of Bianchini's *Algebra* at a November 1992 seminar led by Dr. Anna Słomczyńska at the Centre for Copernican Studies of the Institute for the History of Science. I pointed out the *Algebra* may have been the common stock source for algebraic treatises written by Piero della Francesca and Luca Pacioli.

[1] This was shown by several historians of medieval mathematics at Siena University in Italy, cf. R. Franci and L. Toti Rigatelli, *Fourteenth-century Italian algebra*, in: *Mathematics from Manuscripts to Print 1300–1600*, ed. by C. Hay, Oxford, 1988, pp. 11–29, esp. pp. 19–25.

[2] Piero della Francesca applied square equations to solve banking and trade problems, to calculate proportions of metals in alloys, and to solve regular figures and solids. Cf. Piero della Francesca, *Trattato d'abaco*, ed. and supplied with a Preface by G. Arrighi, Pisa, Domus Galieleana, 1970. Cf. also M. Daly Davis, *Piero della Francesca's Mathematical Treatises. The „Trattato d'abaco" and „Libellus de quinque corporibus regularibus"*, Ravenna [1977]. S. A. Jayawardene, *The „Trattato d'abaco" of Piero della Francesca*, in: *Cultural Aspects of the Italian Renaissance. Essays in Honour of Paul O. Kristeller*, ed. by Cecil H. Clough, Manchester University Press, 1976, pp. 229–243.

The term „abacus" is ambiguous. In antiquity it denoted above all a calculating technique with „abaci". The Renaissance *scuole d'abbaco* introduced the term to mean algebra, the *regole della cosa* it used. The designation of first algebraists, the „cosists", is actually derived from the *regole della cosa.*

[3] In his letter of February 5, 1464, to Regiomontanus, Bianchini said: „Quantum ad regulas algebrae, de quibus comprehendo, vos doctissimum esse, ego quidem in iuventute, dum operationem mercantium operarem, aliquantulum in hoc me delectavi[...]". Ed. by M. Curtze, cf. note 12, p. 238. According to a record from about 1396, one Masolo of Perugia, a teacher at a Venice abbacus school, explained methods to solve some kinds of cubic equations. Cf. R. Franci, L. Toti Rigatelli, op.cit., p. 22. As writes W. Kaunzner: "One question yet to be answered is where Regiomontanus and his contemporary German mathematicians (Aquinas Dacus, Fridericus Gerhard) had actually received their algebraic education." Cf. W. Kaunzner, *Über das Eindringen algebraischer Kenntnisse nach Deutschland*, in: *Rechenpfennige.* Kurt Vogel zum 80. Geburtstag [...] gewidmet von Mitarbeitern und Schülern. Aufsätze zur Wissenschaftsgeschichte, Forschungsinstitut des Deutschen Museums in München, München, 1968, p. 115. Italian influence on algebraic treatises written in German late in the 15th century are discussed in same, cf. p. 99ff.

[4] W. van Egmond, *Practical Mathematics in the Italian Renaissance: A Catalog of Italian Abacus Manuscripts and Printed Books to 1600, Supplemento agli Annali dell'Istituto di Storia delle Scienze di Firenze,* Firezne, 1980, fasc. 1.

[5] R. Franci, L. Toti Rigatelli, *Introduzione all'aritmetica mercantile del Medioevo e del Rinascimento,* Urbino, Quatro venti, 1982, and L. Toti Rigatelli, *Matematici fiorentini del tre-quattrocento,* in: *Symposia Mathematica* vol. 27, 1968, pp. 3–67, Istituto Nazionale de Alta Matematica.

[6] Cf., for example, S. A. Jaywardene, *The Influence of Practical Arithmetics on the Algebra of Rafael Bombelli,* Issis, vol. 64, 1973, pp. 510–523.

[7] J. Sesiano, *On an algorithm for the approximation of surds from a Provençal treatise*, in: *Mathematics from Manuscript to Print*, ed. by C. Hay, Oxford, Clarendon Press, 1988, pp. 30–55, and the same author's *Une arithmétique médievale en langue provençale*, Centaurus, vol. 27, 1984, pp. 26–27.

[8] For latest publications on that see W. van Egmond, *How algebra came to France*, in: *Mathematics from Manuscript to Print*, op.cit., pp. 127–144; G. Flegg, *Nicolas Chuquet – an Introduction*, ibid., pp. 96–116.

[9] In 1145, Robet of Chester, and Englishman working in Spain, translated a textbook of algebra by al-Khwarizmi (Alcharizmi), *Al-kitab al muchtasar fi hisab al-dzhabr wa'l-mukabala*, written about the year 850. Gerard of Cremona produced another translation of the same textbook. A first „adaptation" of al-Khwarizmi's algebra was made by John of Sevilla in the latter half of the 12th century. In the book, called *Liber algorismi de practica arithmeticae*, John presented a discussion of fundamentals of arithmetic, but also tried to solve three kinds of square equations (of the six distinguished by al-Khwarizmi). Early in the 13th century, Leonardo Fibonacci of Pisa (ca. 1179 – after 1240) basing himself on the work of al-Khwarizmi wrote an algebraic treatise called *Liber abaci*, In the above-mentioned line of development of European algebra which seemed to have drawn directly on ancient Greek mathematics, there is the work of Jordanus Nemorarius (fl. between 1230 and 1260), author of treatises in mathematics and mechanics, including an algebraic treatise *De numeris datis*. B. B. Hughes, the editor of the work, contends Nemorarius' treatise was the first significant textbook of algebra produced in Europe since Diophantus' *Algebra* way back in the 3rd century. See B. B. Hughes (ed.), *Jordanus Nemorarius, De numeris datis*, Berkeley, 1981. But Arabs must be credited with developing algebra autonomously in the 11th century (al-Karadji and al-Samwal), cf. R. Rashed, *L'arithmétisation de l'algèbre au 11ème siècle,* in: Proceedings of the 13th International Congress of the History of Sciences, vol. 3–4, Moscow, 1974, pp. 63–69.

[10] Bianchini's other works than the *Flores* are also significant to 15th-century mathematics and astronomy. Bianchini's *Tabulae primi mobilis* were later used, as comparisons had shown, by Regiomontanus to compile his *Tabulae directionum profectionumque*. Cf. G. Rosińska, *Tables trigonométriques de Giovanni Bianchini*, Historia Mathematica, vol. 8, 1981, pp. 46–55. Bianchini's planetary tables (showing motions of planets in latitude) were known to Copernicus who copied them during his Cracow studies; cf. the same author, *Identyfikacja szkolnych tablic astronomicznych Kopernika* (*Identification of Copernicus' scholar planetary tables*), Kwartalnik Historii Nauki i Techniki, vol. 29, 1984, pp. 637–644; the same autor, *Decimal Positional Fractions. Their Use for the Surveying Purposes (Ferrara 1442),* Kwartalnik Historii Nauki i Techniki, vol. 40, 1995, pp. 17-32.

[11] G. Federici Vescovini, *Bianchini Giovanni* (Iohannes Bianchinus. Iohannes de Bianchinis), in: *Dizionario biografico degli Italiani*, vol. 10, 1968, pp. 194–196.

[12] That a study under this title was written by Bianchini was known to 19th-century Italian historians and bibliographers, including Gianmaria Mazzuchelli, Pietro Riccardi, Girolamo Tiraboschi, as well as to German historians of science at the turn of the 19th to the 20th centuries, among them Moritz Kantor and Maximilian Curtze. In many sources, including Bianchini's own astronomical studies and his letters to Regiomon-

tanus of 1463–1464, numerous references to the *Flores* are found. Cf. M. Curtze, *Der Briefwechsel Regiomontans mit Giovanni Bianchini, Jacob von Speier und Cristian Roder*, in: *Urkunden zur Geschichte der Mathematik im Mittelalter und der Renaissance*, 1. Teil, Leipzig, 1902, pp. 192–291. The first edition of the correspondence was published in the 18th century, T. Murr, *Memorabilia Bibliothecarum Publicarum Norimbergensium et Universitatis Altdorfinae*, Pars 1: *Epistolae autographae Johannis Bianchini...*, Norimbergae, 1786.

[13] L. A. Birkenmajer, *Flores Almagesti. Ein angeblich verloren gegangener Traktat Giovanni Bianchini's Mathematikers und Astronomen von Ferrara aus dem XV. Jahrhundert*, Extrait du *Bulletin de l'Académie des Sciences de Cracovie*, Cracovie, 1911. Bianchini's work fills the codex embracing folios 1r-116r. Birkenmajer's description of the contents of the *Flores* refers to the folios 1-50.

[14] Here is Birkenmajer's note concerning „The second treatise" (the *Algebra*) of the *Flores Almagesti* (cf. op.cit., pp. 275–276, and note one therein): „fol. 12 recto: Tractatus secundus Johannis de Bianchinis de demonstrationibus cum Regulis aggregatis. de practica regularum Argebre (sic!) capitulum primum. In tota practica regularum argebre (sic!) quatuor demonstrationibus seu numerorum vocabulis communiter utuntur, scil. Rei, Censui, Cubui et Censui de censu. Res enim idem sonat quantum radix. Census autem quadratum sonat... Es folgt hier die vollständige Lehre von der Transformation und Auflösung der Gleichungen des ersten und zweiten Grades; fol. verso: De practica multiplicandi opportuna (sic!) in regulis argebre (sic!), capitulum 2-um; fol. 13 recto: De practica dividendi opportuna in regulis argebre (sic!), capit. 3-tium; fol. 13 verso – 14 recto: Tractatus secundus incipit. Regule conclusionum ad practicam argebre (sic!) in simplicibus liber secundus... ... Nunc volo te cautum reddre et revelare secretum, quod per alios non revelitur (sic!) ut propter defectum doctrine decipiaris..." L. A. Birkenmajer mentions in a note that this last-quoted sentence is annotated on the margin with the interesting gloss: preter Mahumetum de algebra et almuchabala nec non Johannem de Muris in Quadripartito numerorum et ceteros moderniores. E. Zinner identified the author of the gloss as Regiomontanus.

[15] In the 1950s, Lynn Thorndike discovered five other copies of the *Flores* in libraries in Rome (Biblioteca Vaticana ms. Vat. Lat, 2228 and ms. Vat. Reg. Lat. 1904), Bologna (Biblioteca Universitaria ms. No. 19(293)), Perugia (Biblioteca Palatina ms. 1004), and Paris (Bibliothèque Nationale ms. lat 10253). Cf. L. Thorndike, *Giovanni Bianchini in Paris Manuscripts*, Scripta Mathematica, vol. 16, 1950, pp. 5–12 and 176–180, and *Giovanni Bianchini in Italian Mansucripts*, Scripta Mathematica, vol. 19, 1953, pp. 5–17. For a description of the Paris ms. see E. Poulle, *La bibliotheque scientifique d'un imprimeur humaniste au XV siècle. Catalogue de manuscripts d'Arnaud de Bruxelles à la Bibliotheque Nationale de Paris*, series Travaux d'Humanisme et Renaissance, vol. 57, Génève, 1963, pp. 38–44.

[16] Among the recent publications on Bianchini's achievement we note one by A. Gerl, *Trigonometrisch-astronomisches Rechnen kurz vor Copernicus. Der Briefwechsel Regiomontanus-Bianchini*, Stuttgart, 1989, Boethius, vol. 21. It contains a discussion of how Bianchini avoided cubic equations in his calculation of the sine of 1^{0} (cf. p. 267). Gerl, however, makes no reference to Bianchini's algebraic treatise.

[17] Ms. B 601 ff. 62r-68v; the *Algebra* was copied along with several arithmetic and trigonometric fragments from the *Flores* (ff. 65r-68v). Cf. *Catalogus codicum manuscriptorum medii aevi Latinorum qui in Bibliotheca Jagellonica Cracoviae asservantur*, vol. 4, Wratislawiae, 1986, p. 180.

[18] Bianchini first mentioned the *Flores* to Regiomontanus in a letter dispatched on November 21, 1463, cf. M. Curtze, op.cit., p. 206, then in a letter of February 5, 1464, ibid., p. 241. In reply to the last-mentioned letter written still in February of the same year, as follows from the first sentence of the reply: „Accepi undecima mensis huius Februari litteras vestras expectatissimas...", Regiomontanus goes on to say: „Grandem ingeritis mihi libidinem videndi flores almagesti, quos compilastis, et alia opera vestra" ibid., pp. 242 and 259.

[19] E. Zinner, *Regiomontanus: His Life and Work*, tr. by E. Brown (Studies in the history and philosophy of mathematics, vol. 1) Amsterdam, New York, Oxford, Tokyo 1990, p. 69.

[20] G. Rosińska, *L'audience de Regiomontanus à Cracovie au XVe et au début du XVIe siecle*, Regiomontanus-Studien, ed. by G. Hamann, Wien, 1980. Österreichische Akademie der Wissenschaften, Philosophisch-historische Klasse Sitzungsberichte, vol. 364, pp. 317–326.

[21] L. A. Birkenmajer, *Marcin Bylica z Olkusza oraz narzędzia astronomiczne, które zapisał Uniwersytetowi Jagiellońskiemu w r. 1493* (*Martin Bylica of Olkusz and the astronomical instruments he bequeathed on Jagellonian University in 1493*), Rozprawy Akademii Umiejętności. Wydz. Matem-Przyrod. series 2, vol. 5, Kraków, 1893, pp. 1–164.

[22] In my article *Giovanni Bianchini – matematyk i astronom XV wieku* I cite those notes as anonymous. The catalogue description of the ms BJ 558 in the *Catalogus codicum manuscriptorum medii aevi Latinorum qui in Bibliotheca Jagellonica Cracoviae asservantur*, vol. 3, Wratislaviae, 1984, pp. 384–385, does not attribute the gloss to Regiomontanus either. M. Zwiercan indicates Ferrara as the most likely place where the BJ 558 was produced, see *Catalogus...*, op.cit., p. 385.

[23] Cf. for instance Regiomontanus' hand in the text of the solution of the „third problem" put by Bianchini to Regiomontanus in a November 21, 1463, letter: Quero duos numeros proportionales ut 5 ad 8, quorum ad invicem productus equatur aggregationi ipsorum" (Stadtbibliothek Nürnberg, ms Cent 5 app 56^c, f. 23r), and *ibid.*, also in Regiomontanus' hand, the following text: Quartum interrogatum. Divisi 10 in duos quorum maiorem per minorem divisi iterum... That is a draft version of a letter Regiomontanus sent Bianchini at the turn of 1463 to 1464 at the latest. Bianchini's reply to it was dispatched on February 5, 1464. Two folios, 23r and 26r, from the ms. Cent 5 app 56^c were reproduced by W. Kaunzner, *Über Regiomontanus als Mathematiker*, ibid., pp. 125–145, Plates Nos. 5 and 6.

[24] Cf. *Catalogus...*, op.cit., vol. 4, Wratislawiae, 1985, pp. 176–192. Pages 180–181 refer to different portions of the *Flores*, including the *Algebra*; in the same place, the treatise was identified as a work of Bianchini's. For the establishment of the manuscript's Italian provenience and its dating after 1474 see pp. 190–191. The catalogue description of the ms is signed E.I.

[25] Martin Biem of Olkusz graduated as magister from Cracow University in 1491, shortly before Copernicus enrolled. Cf. L. A. Birkenmajer, *Martini Biem de Olkusz Poloni Nova Calendarii Romani Reformatio* (Introduction), Cracoviae, 1918. A. Birkenmajer, *Biem Marcin*, in: Polski Słownik Biograficzny, vol. 2, Cracow, 1936, pp. 68–69.

[26] Mikołaj Młodszy of Wieliczka, called Mleczko, got his bachelor's degree at Cracow University in 1508 and his master's degree in 1513. He studied medicine in Bologna in 1514 to 1516 (doctorate on August 16, 1516). Professor of medicine at Cracow University from 1518 on. He died in 1519. Cf. H. Barycz, *Historia Uniwersytetu Jagiellońskiego w epoce humanizmu* (*History of Cracow University in the age of humanism*), Kraków, 1935, pp. 231–232.

[27] The emergence of geometrical algebra is usually considered as an effect of efforts to overcome the crisis of mathematics (5th century BC) following the discovery of incommensurable quantities. Cf., for instance, *Historia matematyki*, ed. by A. Juszkiewicz, vol. 1: *Od czasów najdawniejszych do początków czasów nowożytnych* (*From earliest times to the beginnings of modern times*), translated from Russian by S. Dobrzycki, Warsaw, 1975, ch. 4; I. G. Baszmakowa, *Grecja starożytna* (*Ancient Greece*), p. 86.

[28] Bianchini, *Algebra*. In tota practica regularum algebrae quatuor denominationes seu quatuor vocabula cummuniter utuntur, scilicet res, census, cubus and census de censu. Res enim idem sonat quantum radix, census autem quadratum sonat seu superficiem quadratam, cubus vero corpus solidum. Census de censu est quadratus quadrati; que omnia a radice seu a re oriuntur. The quoted fragments of the *Algebra* are taken from the tipe-script of the above-mentioned critical text now in preparation for print.

[29] G. l'Huiller compared a copy of the algebra of Jean de Murs, a 14th-century mathematician and astronomer, with the accompanying notes by Regiomontanus (who was keen to get the manuscript published). According to L'Huillier „Regiomontanus rejette souvent la solution purement algébrique des équations que Jean de Murs tire du Liber abaci de Léonard de Pise..." G. I. l'Huillier, *Regiomontanus et le Quadripartium numerorum de Jean de Murs, Revue d'Histoire des Sciences, vol. 33, 1980, p. 197.*

[30] Here is Bianchini's definition of the rule of signs: Quando plus multiplicatur per plus productus erit plus et hoc clarum est. Quando plus multiplicatur per minus aut minus per plus productum erit minus et hoc patet, quia quantum minus augetur aut plus minuetur tantum productum fiet minus. Quando minus multiplicatur per minus productus erit plus, quia quantum minus minuetur tantum plus augetur. See also G. Rosińska, *A Chapter in the History of the Renaissance Mathematics: Negative Numbers and the Formulation of the Law of signs (Ferrara, Italy ca. 1450).* Kwartalnik Historii Nauki i Techniki 40 (1995), pp. 3–20.

[31] The exercises were preserved only in two of the seven extant codices containing Bianchini's *Algebra*, namely the BJ 601, ff. 63v-64r, and BN lat. 10253, ff. 29r-30v.

[32] Here are the examples in modern notation: 1. $ax=c$, $x=c/a$.

2. $ax^2=c$, $x=\sqrt{c/a}$. 3. $ax^3=c$, $x=\sqrt[3]{c/a}$. 4. $ax^4=c$, $x=\sqrt[4]{c/a}$.

5. $ax^2=bx$, $x=b/a$. 6. $ax^3=bx$, $x=\sqrt{b/a}$. 7. $ax^4=bx$, $x=\sqrt[3]{b/a}$.

8. $ax^3=bx^2$, $x=b/a$. 9. $ax^4=bx^3$, $x=b/a$. 10. $ax^4=bx^3$, $x=b/a$.

[33] For the aapearance of negative solutions in European mathematics see J. Sesiano, op.cit., pp. 116–119.

[34] Nunc volo te cautum reddere et revelare secreta quae per alios non revelantur, ne propter defectum doctrinae ab aliis decipiaris. Quare nota bene et memoriae commenda quod quando quadratum medietatis rerum non excederet numerum cum censu datum, positio erit impossibilis, nec super ipsam oportet laborare. Si vero erit aequalis numero dato, tunc medietas rerum absque alia diminutione seu additione valet rem [$x=b/2$]. Saepenumero etiam contingit quod duplici modo respondere possumus, puta in propositione suprascripta [case V] Videlicet dato quod unus census et 24 numeri aequantur 14 rebus [$x^2+24=14x$] dico quod debemus, ut

supra, mediare res et medietatem in se multiplicare et de producto subtrahere numerum; cuius radix addita medietati rerum valet res [$x = b/2 + \sqrt{(b/2)^2 - c}$]. Et hoc quia possum ponere censum maiorem quadrati medietatis radicum [...].

[35] De regulis adhaerentibus primae regulae de compositis. Capitulum secundum. Quando res aequantur censibus et cubis debemus partes reducere ad unum cubum, id est dividere per cubos, deinde mediare census et medietatem in se multiplicare et productum addere rebus, cuius aggregati radix diminuta medietate census valet rem. Quando census aequantur cubis et censibus de censu debemus partes, ut supra, reducere ad unum censum de censu, deinde mediare cubos et aggregati radix diminuta medietate cuborum valet rem.

[36] Studies of algebra in Germany, as by W. Kaunzner, *Über das Eindringen algebraischer Kenntnisse,* op.cit., pp. 91–122, and studies occasioned by the Regiomontanus anniversary in 1976, as W. Kaunzner, Über *Regiomontanus,* op.cit., pp. 125–145, or M. Folkerts, *Die mathematischen Studien Regiomontans in seiner Wiener Zeit,* in: op.cit., pp..175–209, provided only a partial answer.

[37] L. A. Birkenmajer, *Marcin Bylica z Olkusza,* op.cit., pp. 57, 144, notes 238, 241.

[38] Regiomontanus must have got manuscript BJ 558 in his hands for a first time not later than in February 1464. The codex Vat. Lat 2228 was completed in Ferrara on December 4, 1470. Manuscript BJ 601 was not produced before 1474. The copying of the Pais manuscript, BN. Lat 10253, was completed by Arnold of Brussels in Naples on January 21, 1481 (first five treatises of the Flores) and on March 8, 1487 (the next three treatises). Analysis of the writing of the three undated codices: Bologna BU 19(293), Perugia B. Palatina No. 1004, and Vatican Vat. Reg. Lat. 1904, indicate the last decade of the 15th century as a likely date of production.

[39] See the description of ms. BJ 558, in *Catalogus...* ut supra, vol. 3, p. 386.

[40] See G. Rosińska, *Scientific Writings and Astronomical Tables in Cracow. A Census of Manuscript Sources (XIVth–XVIth Centuries),* Wrocław, 1984, Nos. 44, 28, 63, 123, 218, 298, 425, 429, 446, 485, 708, 709, 1128, 1129, 1131–1134, 1220, 1451, 1598, 1659, 1660, 1867, 2136, 2202, 2383.

[41] J. Dianni, *Pierwszy znany traktat rękopiśmienny w literaturze matematycznej w Polsce. Algorismus minutiarum Martini Regis de Premislia (A first know mansucript treatise in the mathematical literature of Poland),* Kwartalnik Historii Nauki i Techniki, vol. 12, 1967, pp. 269–280.

[42] Preserved in the Jagellonian Library, ms. BJ 1927, f. 250r-318r, and in the Bodleian Library, Oxford, ms. Can. misc., 499, f. 212r-249r.

[43] G. Rosińska, *Scientific Writings,* op.cit., p. 548.