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Stanisław Jaśkowski's Logical Investigations

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1. Introduction

Stanisław Jaśkowski was a mathematician by education. His interests focused on logic and foundations of mathematics. Jaśkowski's contribution in the world of logic concerned both classical and non–classical calculi. As regards classical logic, he is mostly known for development of a system of natural deduction. He analyzed also parts of Aristotle's calculus of names from the point of view of modern mathematical logic. These were historical studies which were not so important as his discovery and formulation of a method of natural deduction. As regards non–classical logic, the most important studies are those on the intuitionistic propositional calculus and his suggestion of building a discursive logic, which was the first attempt at constructing system *tolerating* contradictory opinions. Jaśkowski's system was the first logical paraconsistent system. Jaśkowski also provided a basis for the development of causal logic based on propositional calculus with dependent sentential variables.

The purpose of this paper is to present an outline of Jaśkowski's logical achievements. As his theories are often not intuitive, formal notations are changed. Jaśkowski used Łukasiewicz's notation. We will use bracket symbolism and supply the necessary comments. These will assist in understanding the formal notations proposed by Jaśkowski. This paper has a mainly historical nature and belongs to the history of logic, not to logic in the strict sense.

2. The history of logic – an interpretation of Aristotelian categorical sentences

Representatives of the Lvov–Warsaw school, who include Jaśkowski, turned many times to the old books. The object of their analysis, often narrow in scope, was Aristotle's logic[1]. It seems quite understandable because Aristotle is widely acknowledged as the founder of logic. He was the first to used variables, developed the analytical syllogism, and analyzed the semantic properties of language. The calculus of names is associated with some semantic problems.

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[1] For example J. Łukasiewicz, *O zasadzie niesprzeczności u Arystotelesa*, Warszawa 1910, J. Łukasiewicz, *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, Oxford 1951.

Jaśkowski, like his master, Jan Łukasiewicz, studied Aristotle's logic, mainly from a semantic point of view. He studied interpretations of categorical sentences. Aristotle did not consider in his logic substitutions of empty names and the most general names – universal names. The empty and universal name can not be taken as terms in Aristotelian syllogisms.

Some modes fail when empty names are substituted. Names which were not used by Aristotle in his logic were defined by Jaśkowski as non–Aristotelian names. He indicated that these names have many applications. They may be used to formulate sentences which deny existence: *none object is a centaur.* In each modern calculus of names, where it is allowed to substitute non–Aristotelian names for nominal variables, some rules of classical theory of categorical inference were rejected and replaced by weaker theorems, e.g. weakened by adjoining a additional existential premise, as Brentano did[1].

Scholars give different meanings to categorical sentences. Kotarbiński, for example, gives two meanings for general affirmative sentence. First, he introduces the strong general affirmative sentence. We understand the sentence "each A is B" in this way that for each x, if x is A, then x is B and for some x, x is A. This means that the class of objects A is included in class of objects B and A is not an empty name. Secondly, a weaker interpretation may be given to the general affirmative sentence. The expression "each A is B" may be understood in this way: "for each x, if x is A, then x is B" or in other words "if anything is A, it is also B". In the case of the first understanding of the sentence of function "each... is...", if we substitute the empty name *centaur* for A, and *quadruped* for B, we obtain the false sentence *each centaur is a quadruped and some objects are centaurs.* In the case of the second understanding of this affirmative sentence, we will receive the true sentence *each centaur is a quadruped*[2].

If affirmative general sentences and other sentences called *categorical sentences* can be thus homogenously understood, are all classical inference principles, valid for Aristotelian names, valid for non–Aristotelian names? This problem was still not resolved. Jaśkowski put in this context the following problem: is it possible in a calculus of predicates to define a system of relations between predicates which would constitute new meanings (interpretations) of the four categorical traditional sentences and name negation in such way that all classical theory rules are valid? It turned out that the answer is affirmative, and there is several interpretations with this property. Only one of these interpretations corresponds with the meaning of categorical sentences for Aristotelian names. In this interpretation we treat the empty name, as an intersection of empty denotation (empty set) with denotation of any Aristotelian name[3].

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[1] Cf. S. Jaśkowski, *O interpretacjach zdań kategorycznych Arystotelesa w rachunku predykatów*, p. 78.

[2] Cf. T. Kotarbiński, *Elementy teorii poznania, logiki formalnej i metodologii nauk*, [4th ed.] Warszawa 1990, pp. 211–212.

[3] Cf. S. Jaśkowski, *O interpretacjach zdań kategorycznych Arystotelesa w rachunku predykatów*, p. 78.

Jaśkowski used the method of contemporary mathematical logic. Strictly speaking, he formulated in the lower calculus of predicates the point at issue. Jaśkowski takes the traditional notation. Variables *S*, *P* denotes names, *P'* denotes negation of *P*. Metalogical variable α represents formulae which may contain symbols *a*, *e*, *i*, *o*, ‘, the nominal variables and function symbols of the sentential calculus. Jaśkowski defines in the predicate calculus relations between unitary predicates: $Sa_x P$, $Se_x P$, $Si_x P$, $So_x P$ and the function $S = P'_x$. Each sentence containing the functions a_x, e_x, i_x, o_x, $'_x$, is said to be a transcription of *X* where $X = B, E, J, K$. The expression α_x denotes formula of calculus of predicates. The expression α_x is obtained from α by means of the substitution of symbols a_x, e_x, i_x, o_x, $'_x$, respectively for *a*, *e*, *i*, *o*, ‘. The predicates are denoted by the same letters as nominal variables in traditional theory[1]: $P_1, P_2, P_3, \ldots, S_1, S_2, S_3, \ldots M$, the sign “$\vdash$” is the sign of assertion.

In the first place Jaśkowski defines concept of Aristotelian name, designated as $Ar(P)$ (read: *P* is the Aristotelian name)[2]:

(1) $Ar(P) = \exists x\, P(x) \wedge \exists x \sim P(x)$

P is an Aristotelian name if some objects are *P* and some objects are not *P*.

(2) $Ar\,(P_1 \ldots P_n) = Ar(P_1) \wedge Ar(P_2) \wedge \ldots \wedge Ar(P_n)$

The sequence of names is called Aristotelian if each name in this sequence is an Aristotelian name. $P_1 \ldots P_n$ are not empty or universal predicates.

Then Jaśkowski gives four transcriptions of the categorical sentences. He presents Brentano's transcription *B* first:

(3) $Sa_BP = \forall x\, Sx \rightarrow Px$

The second interpretation is named *equivalence transcription*, symbolically *E*.

(4) $Sa_E P = Sa_BP \wedge Pa_BS$

The third transcription has non Aristotelian predicates, symbolically *J*.

(5) $Sa_JP = [Ar\,(S, P) \rightarrow Sa_B\, P] \wedge [\sim Ar\,(S, P) \rightarrow Sa_E\, P]$

The short *K* means *transcription reverse to (5)*. The transcription reverse to (5) has the form:

(6) $Sa_K\, P = Pa_J\, S$

For $X = B, E, J, K$:

(7) $P'_x(x) = \sim P(x)$

(8) $Se_x\, P = Se_x\, P'_x$

(9) $Si_x\, P = \sim Se_x\, P$

(10) $So_x\, P = \sim Sa_x\, P$

Jaśkowski assumes that *U* is the meaningful expression of the predicate calculus. *U* only includes predicate variables $P_1 \ldots P_n$. The expression $Ar(P_1 \ldots P_n) \rightarrow U$ is denoted as $\mathrm{Ar} \Rightarrow U$.

W is a tautology of classical theory (notation $\bullet \vdash \mathbf{W}$) if and only if $\vdash \mathrm{Ar} \Rightarrow \mathbf{W}_{\mathrm{B}}$, i.e. $\mathrm{Ar} \Rightarrow \mathbf{W}_{\mathrm{B}}$ is a tautology of predicate calculus[3].

[1] The Aristotelian theory of categorical sentences is called the classical theory.

[2] S. Jaśkowski, *O interpretacjach zdań kategorycznych Arystotelesa w rachunku predykatów*, p. 80.

[3] S. Jaśkowski, *O interpretacjach zdań kategorycznych Arystotelesa w rachunku predykatów*, p. 80.

Jaśkowski considered the following conditions:
(a) for every **W** if • $\vdash$ **W**, then $\mathbf{W}_x$
(A) for every **W**, • $\vdash$ **W** if and only if $\mathbf{W}_x$
(b) $\vdash \mathrm{Ar}(P) \to \mathrm{Ar}(P'_x)$
(c) for every W, $\vdash \mathrm{Ar} \Rightarrow \mathbf{W}_x \leftrightarrow \mathbf{W}_B$

Jaśkowski proved that conditions (a) and (b) are simultaneously satisfied, if and only if $X = E$ or $X = J$ or $X = K$; conditions (A) (b) are simultaneously satisfied, if and only if X = J or X = K; The interpretation *J* satisfies three conditions: (a) (b) (c)[1].

Interpretations of categorical sentences equivalent to Jaśkowski's transcriptions permit the presentation of traditional logic as part of contemporary classical logic, without the introduction of axioms from outwith logic. But this interpretation distorts the traditional and common sense of categorical sentences.

3. Research on the classical sentential calculus. The natural deduction calculus

3. 1. There was a trend in research into the sentential calculus in the interwar period to minimize the number of propositional calculus axioms. We may say that the natural deduction calculus is a result of these minimization trends which led Jaśkowski to discover the sentential calculus which was not based on any axiom. Jaśkowski's research was motivated also by Łukasiewicz[2], who searched for a way, strictly speaking a logical system, in which it would be possible to conduct inference simulating a mathematician's reasoning. Jaśkowski built a logic system based on natural deduction rules. He announced results of his studies in 1927, at the Congress of Polish Mathematicians in Lvov. The results he obtained made a fundamental contribution to the development of logic. The paper on this topic was published in 1934, in the first issue of the journal *Studia Logica*[3] (established by Łukasiewicz), seven years after the first shared them. This is important, as in 1934 Gentzen published his article on natural deduction[4]. *Mathematische Zeitschrift*, where the results of Gentzen's studies were published, had in those times a incomparably wider distribution than the just established *Studia Logica*. The natural deduction system is called a *Gentzen system* and he is commonly regarded as the founder of natural deduction calculus.

Natural deduction is one of two methods of construction of deductive systems in logic. We may formally write that each deductive system of logic is a pair $\langle A, R\rangle$, where *A* is a set of axioms, and *R* is the set of deductive rules. If *A* is the empty set, then this is the natural deduction system.

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[1] S. Jaśkowski, *O interpretacjach zdań kategorycznych Arystotelesa w rachunku predykatów*, pp. 80–87.

[2] L. Jeśmianowicz, *Stanisław Jaśkowski*, p. 131.

[3] S. Jaśkowski, *On the Rules of Suppositions in Formal Logic*.

[4] G. Gentzen, *Untersuchungen über das logische Schliessen* in: *Mathematische Zeitschrift* 39, 1944, pp. 176–210 & pp. 405–431.

The natural deduction system may be presented in modified form. The simple mathematical proof has the following form[1].

The variables $x_1, x_2, y_1, y_2, z, e, m, n$, represent integers.

We have the theorem:

If x_1, x_2, are divisible by z, then $x_1 \times y_1 + x_2 \times y_2$ is divisible by z.

The proof of this theorem is as follows:

(1) x_1, x_2, are divisible by z supposition

On the base of definition of divisibility:

x / y if and only if $\exists m \in C\ (x = y \times m)$

we receive that there are such integers m_1, m_2, that:

(2) $x_1 = m_1 \times z$

$x_2 = m_2 \times z$

Therefore we infer

(3) $x_1 \times y_1 = m_1 \times y_1 \times z$

$x_2 \times y_2 = m_2 \times y_2 \times z$

From this we obtain:

(4) $x_1 \times y_1 + x_2 \times y_2 = m_1 \times y_1 \times z + m_2 \times y_2 \times z$

(5) $x_1 \times y_1 + x_2 \times y_2 = (m_1 \times y_1 + m_2 \times y_2) \times z$

So, it results on the basis of definition of divisibility that:

$x_1 \times y_1 + x_2 \times y_2$ is divisible by z, what was to be proved.

The above proof is a simple mathematical proof in which the following steps can be distinguished: we perceive that a thesis to be proved has a conditional form. We take its antecedent as a premise, from which conclusions are derived by rules, definitions and laws earlier accepted. We proceed in proving to reach a consequent of the proved thesis. So, if we, on p base, reach q, we can write that if p, then q.

Jaśkowski proceeded similarly analyzing proofs that use the natural deduction method. He searched for a formal rule which allows to belief that the proposed thesis is true. Let's take formula $p \rightarrow (p \rightarrow q) \rightarrow q$. Let's suppose p. We may write it as Sp. "S" letter symbolizes supposition. We suppose $p \rightarrow q$. Therefore, q results from p and $p \rightarrow q$. Thus, we perceive that q is a consequence of the supposition $p \rightarrow q$. We obtain, as deductive conclusion, that if p implies q, then q i.e. $p \rightarrow q \rightarrow q$. In this way, having supposed p, we have deduced sentence $p \rightarrow q \rightarrow q$. Therefore, we can infer $p \rightarrow (p \rightarrow q) \rightarrow q$[2].

For the above process of deduction to be more clear, Jaśkowski introduces the so–called prefixes, indicating which sentences are consequences of the given supposition. Prefixes with specific numbers relate to individual suppositions. Number "1" will relate to the first supposition, p in the above example, so we will write "$1{\cdot}Sp$" (we suppose p). There will be expression $p \rightarrow q$ within premise p, of which prefix will also include number 1. However, since $p \rightarrow q$ is premise itself, we provide it additionally with its own number written after 1: "$1{\cdot}1{\cdot}S\ p \rightarrow q$" (we assume that if p, then q). The expression

[1] L. Borkowski, J. Słupecki, *Elementy logiki matematycznej i teorii mnogości*, Warszawa 1984, pp. 10–11.

[2] S. Jaśkowski, *On the Rules of Suppositions in Formal Logic*, p. 6.

$p \rightarrow q$ also makes supposition within the first supposition. The sentence q infers from Sp and $Sp \rightarrow q$. On the base of $p \rightarrow q$ we obtain q, then $p \rightarrow q \rightarrow q$. From Sp, we can infer $p \rightarrow (p \rightarrow q) \rightarrow q$, which is not provided with any number. Jaśkowski presents this sequence formally in the following way:[1]

$1{\cdot}Sp$
$1{\cdot}1{\cdot}S\, p{\rightarrow}q$
$1{\cdot}1{\cdot}q$
$1{\cdot}p \rightarrow q \rightarrow q$
$p \rightarrow (p \rightarrow q) \rightarrow q$

The procedure applied in the case of supposition leading to contradiction.

$2{\cdot}S \sim p \rightarrow \sim q$
$2{\cdot}1{\cdot}Sq$
$2{\cdot}1{\cdot}1{\cdot}S \sim p$
$2{\cdot}1{\cdot}1{\cdot}{\sim}q$
$2{\cdot}1{\cdot}p$
$2{\cdot}q \rightarrow p$

We assume that $\sim p \rightarrow \sim q$. We can assume negation of the consequent of the supposition 2, i.e. q within this supposition. The next supposition is included within supposition 2·. This is an antecedent of $\sim p \rightarrow q$, i.e. $\sim p$. From $S{\sim}p \rightarrow \sim q$ and $\sim p$ we obtain $\sim q$. The supposition of $\sim p$ with prefix 2·1·1· leads to the contradiction of q and $\sim q$. Therefore p infers from supposition q in 2·1·. We can write $q \rightarrow p$ and provide it with prefix 2·. From $S{\sim}p \rightarrow \sim q$ infers q → p. From it follows $\sim p \rightarrow \sim q \rightarrow (q \rightarrow p)$.

On the basis of the above considerations Jaśkowski begins to formulate rules of his system. According to him all steps of proof e.g. steps written in above examples, are theses of the system and no other theses exist. For example, the thesis of the form "2·1·*Sq*" and all theses having their initial parts equiform with the prefix "2·1·" i.e. theses "2·1·1·*S~p*", "2·1·1·*~q*", "2·1·*p*" make up the class of supposition α and of all expressions which in other theses are preceded by initial parts equiform to the prefix of α. Jaśkowski called this thesis *the domain of the supposition* α. He gave the name *domain* to the class of all theses belonging to the system as well as the domain of suppositions. The domain will be a set of theses which are written down to a given moment. It is conception of deductive system as a developing system[2]. If a system does not include any thesis, then a domain will be the empty set. The expression can be included in domain D, if it satisfies some condition F. Jaśkowski made the following rules[3]:

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[1] S. Jaśkowski, *On the Rules of Suppositions in Formal Logic*, p. 7.

[2] S. Jaśkowski, *On the Rules of Suppositions in Formal Logic*, p. 9.

[3] S. Jaśkowski, *On the Rules of Suppositions in Formal Logic*, pp. 10–11. In seventies Orłowska gave formal rules for Jaśkowski's natural deduction. See E. Orłowska, *On the Jaśkowski's method suppositions*, p. 189.

Rule 1. We can attach to every domain D an expression composed of (1) a number, which is not equiform with the initial number of any other element of domain D, (2) a dot, (3) a symbol "S", (4) a sentence. According to this rule, we may write formulas: "$1 \cdot Sp$", "$1 \cdot 1 \cdot Sp \rightarrow q$".

Rule 2. If in the domain D of a supposition α sentence β is true, we can join a sentence of the form $\alpha \rightarrow \beta$ to a domain, for which D is an immediate sub–domain. This rule allows to acknowledge the expression with form "$1 \cdot p \rightarrow q \rightarrow q$" due to "$1 \cdot 1 \cdot Sp \rightarrow q$" and "$1 \cdot 1 \cdot q$".

Rule 3. In the domain D, there are sentences composed of a symbol (1) "$\rightarrow$", (2) a sentence α (3) a sentence β. If $\alpha \rightarrow \beta$ is true, and α is true, then we can add to D sentence equiform with β. For example, conclusion "$1 \cdot 1 \cdot q$" is inferred from "$1 \cdot Sp$" and "$1 \cdot 1 \cdot Sp \rightarrow q$".

Rule 4. If in domain D of supposition of $\sim\alpha$, sentences β and $\sim\beta$ will be true, then we can join to domain D sentence equiform with α for which D is a immediate sub–domain. For example, conclusion "$2 \cdot 1 \cdot p$" infers from premises "$2 \cdot 1 \cdot S \sim p$", "$2 \cdot 1 \cdot Sq$" and $2 \cdot 1 \cdot 1 \cdot \sim q$".

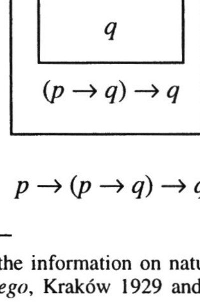
Jaśkowski made on the base of rules 1–4 his system without any axioms. Below I give 51 theses which Jaśkowski obtained by means of the natural deduction method[1]:

(1) $1 \cdot Sp$	R1
(2) $1 \cdot 1 \cdot Sp \rightarrow q$	R1
(3) $1 \cdot 1 \cdot q$	R3, (2), (1)
(4) $1 \cdot p \rightarrow q \rightarrow q$	R2, (2), (3)
(5) $p \rightarrow (p \rightarrow q) \rightarrow q$	R2, (1), (4)
(6) $2 \cdot S {\sim} p \rightarrow {\sim} q$	R1
(7) $2 \cdot 1 \cdot Sq$	R1
(8) $2 \cdot 1 \cdot 1 \cdot S {\sim} p$	R1
(9) $2 \cdot 1 \cdot 1 \cdot {\sim} q$	R3, (6), (8)
(10) $2 \cdot 1 \cdot p$	R4, (8), (7), (9)
(11) $2 \cdot q \rightarrow p$	R2, (7), (10)
(12) ${\sim} p \rightarrow {\sim} q \rightarrow (q \rightarrow p)$	R2, (6), (11)
(13) $1 \cdot 2 \cdot Sq$	R1
(14) $1 \cdot q \rightarrow p$	R2, (13), (1)
(15) $p \rightarrow (q \rightarrow p)$	R2, (1), (4)
(16) $1 \cdot 3 \cdot S {\sim} p$	R1
(17) $1 \cdot 3 \cdot 1 \cdot S {\sim} q$	R1
(18) $1 \cdot 3 \cdot q$	R4, (17), (1), (16)
(19) $1 \cdot {\sim} p \rightarrow q$	R2, (16), (18)
(20) $p \rightarrow ({\sim} p \rightarrow q)$	R2, (1), (19)
(21) $3 \cdot Sp \rightarrow q$	R1
(22) $3 \cdot 1 \cdot Sq \rightarrow r$	R1
(23) $3 \cdot 1 \cdot 1 \cdot Sp$	R1
(24) $3 \cdot 1 \cdot 1 \cdot q$	R3, (21), (23)
(25) $3 \cdot 1 \cdot 1 \cdot r$	R3, (22), (24)

[1] S. Jaśkowski, *On the Rules of Suppositions in Formal Logic*, pp. 12–13.

(26) $3 \cdot 1 \cdot p \rightarrow r$	R2, (23), (25)
(27) $3 \cdot q \rightarrow r \rightarrow (p \rightarrow r)$	R2, (22), (26)
(28) $p \rightarrow q \rightarrow (q \rightarrow r) \rightarrow (p \rightarrow r)$	R2, (21), (27)
(29) $4 \cdot Sp \rightarrow (q \rightarrow r)$	R1
(30) $4 \cdot 1 \cdot Sp \rightarrow q$	R1
(31) $4 \cdot 1 \cdot 1 \cdot Sp$	R1
(32) $4 \cdot 1 \cdot 1 \cdot q \rightarrow r$	R3, (29), (31)
(33) $4 \cdot 1 \cdot 1 \cdot q$	R3, (30), (31)
(34) $4 \cdot 1 \cdot 1 \cdot r$	R3, (32), (33)
(35) $4 \cdot 1 \cdot p \rightarrow r$	R2, (31), (34)
(36) $4 \cdot p \rightarrow q \rightarrow (p \rightarrow r)$	R2, (30), (35)
(37) $p \rightarrow (q \rightarrow r) \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$	R2, (29), (36)
(38) $5 \cdot S{\sim}p \rightarrow p$	R1
(39) $5 \cdot 1 \cdot S{\sim}p$	R1
(40) $5 \cdot 1 \cdot p$	R3, (38), (39)
(41) $5 \cdot p$	R4, (39), (40)
(42) ${\sim}p \rightarrow p \rightarrow p$	R2, (38), (41)
(43) $6 \cdot Sp \rightarrow q \rightarrow p$	R1
(44) $6 \cdot 1 \cdot S{\sim}p$	R1
(45) $6 \cdot 1 \cdot 1 \cdot Sp$	R1
(46) $6 \cdot 1 \cdot 1 \cdot {\sim}p \rightarrow q$	R3, (20), (45)
(47) $6 \cdot 1 \cdot 1 \cdot q$	R3, (46), (44)
(48) $6 \cdot 1 \cdot p \rightarrow q$	R2, (45), (47)
(49) $6 \cdot 1 \cdot p$	R3, (43), (49), (44)
(50) $6 \cdot p$	R4, (44), (49), (44)
(51) $p \rightarrow q \rightarrow p \rightarrow p$	R2, (43), (50)

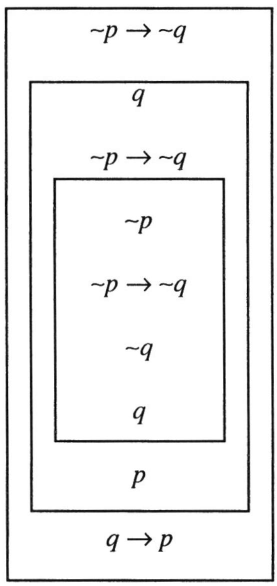
3. 2. In 1926 Jaśkowski expressed his system in a different symbolism. He did not use numerical prefixes, but graphic imaging. We know this from a short note in a text Jaśkowski wrote in 1934[1].



$$p \rightarrow (p \rightarrow q) \rightarrow q$$

[1] For the first time Jaśkowski gave the information on natural deduction system in: *Księga pamiątkowa pierwszego polskiego zjazdu matematycznego*, Kraków 1929 and in note in: *On the Rules of Suppositions in Formal Logic*.

The rectangles indicate the range of suppositions. The suppositions are written under the top sides of the rectangles. The sentences resulting from the suppositions are written along the bottom edges of the relevant rectangles. So in rectangle of the largest surface area, supposition is below the upper side, and sentence which can be derived from this supposition is above its lower side. We can thus write a new sentence, which is not within the interior rectangle area, below the lower side. The expression "p" which have been written outwith the smaller rectangle have been repeated inside them. The second example above has the form:



$\sim p \to \sim q \to (q \to p)$

Let's try to give rules for this symbolism analogous to those for R1–R4[1].

Rule 1'. To every domain *D*, which is symbolized by some rectangle, we can join (1) an expression inside a rectangle; this rectangle is not the same one as the rectangle of initial positions of any other element of domain *D*, (2) the sentence that is below upper side of rectangle. (The sides of rectangles have the same role as dots. One dot means the same as the largest size rectangle, two dots have mean the same as the rectangle which is smaller than the largest but larger than others etc. The position of a sentence below upper side of the rectangle corresponds to supposition *S* in latter a system. The position of the sentence above the lower side of the rectangle in latter system corresponds to the sentence which is an immediate result of supposition.)

[1] In 1934 Jaśkowski put forward only graphical form of his theory from 1926. Rules 1'–4' are an attempt to show that earlier theory demands the same rules as later theory. The rules of expressions' construction and notation's rules are different. However, we can say that rule 1' corresponds to rule 1.

Rule 2'. If in the domain D of supposition α (written below the upper side of a rectangle), sentence β is true, we can add a sentence of the form $\alpha \to \beta$ to a domain of which D is an immediate sub–domain (i.e. we write below the lower side of the same rectangle the sentence $\alpha \to \beta$). For example:

$$\boxed{\begin{array}{c} p \\ \\ q \end{array}}$$

$$p \to q$$

Rule 3'. In domain D, in which sentence α written below the upper sides of the rectangles is true, and the sentence composed of a symbol (1) "$\to$", (2) a sentence α, (3) sentence β, is true, we can add the sentence equiform β. The second sentence in form $\alpha \to \beta$ is written below upper side of rectangle symbolizing immediate subdomain of D. We can add upper lower side of rectangle symbolizing immediate subdomain sentence equiform with β. For example

$$\boxed{\begin{array}{c} p \\ \boxed{\begin{array}{c} p \to q \\ \\ q \\ \\ \end{array}} \\ \\ \end{array}}$$

Rule 4'. If in domain D, assuming of $\sim\alpha$, sentences β and $\sim\beta$ are true, then we can add α to the domain D in an immediate sub–domain. For example

$$\boxed{\begin{array}{c} \sim p \\ \\ q \\ \\ \sim q \end{array}}$$

$$p$$

It is visible that rules 1 and 1' are different because the two languages are constructed in different ways. The rules 2 and 2', 3 and 3', 4 and 4' are analogical. This shows that there was a theory of natural deduction ready in the nineteen twenties.

3. 3. Jaśkowski analyzed also the classical axiomatic sentential calculus. He gave a complete system of axioms for the classical sentential calculus. The set of axioms was based on more than two primitive terms:[1]

(1) $p \vee q \leftrightarrow (p \rightarrow q) \rightarrow q$
(2) $p \rightarrow q \rightarrow [q \vee (p \rightarrow r)]$
(3) $(p \leftrightarrow q) \rightarrow (p \rightarrow q)$
(4) $(p \leftrightarrow q) \vee (q \vee p)$
(5) $p \rightarrow [(q \leftrightarrow p) \leftrightarrow q]$
(6) $p \rightarrow [(p \wedge q) \leftrightarrow q]$
(7) $p \rightarrow [q \rightarrow (r \rightarrow p)]$
(9) $(p \leftrightarrow \sim p) \rightarrow q$
(10) $(p \wedge q) \rightarrow p$

Jaśkowski also gave axioms for the fragmentary sentential calculi. These calculi have sets of axioms which are equal to the sets of tautologies of the classical sentential calculus. The fragmentary sentential calculi contain respective symbols of functions: $\vee$, $\rightarrow$; $\wedge$, $\rightarrow$, $\sim$; $\wedge$, $\rightarrow$, $\leftrightarrow$; $\vee$, $\rightarrow$, $\leftrightarrow$, $\sim$; $\vee$, $\rightarrow$, $\leftrightarrow$, $\wedge$[2]. In the fragmentary sentential calculus, we can define all symbols of functions of classical calculus by means of using not all but some symbols.

4. Studies on the intuitionistic sentential calculus[3]

The Heyting's intuitionistic logic rejects the law of the excluded middle. So this logic is not a normal two–valued logic. Brouwer rejected the law of the excluded middle as it is impossible to demonstrate its absolute truth. This law is also not, in his opinion, false. He maintained that it is false to think that law of the excluded middle is false. This theorem is known in the literature as the law of absurdity of absurdity of the excluded middle. It can be presented as follows:

$$\sim\sim(p \vee \sim p)$$

So one can say that the law of the excluded middle has as it were a third logical value though not in the same sense as in Łukasiewicz's three–valued logic.

The matrix for negation is as follows in classical logic:

p	$\sim p$
1	0
0	1

[1] S. Jaśkowski, *Trois contributions au calcul des propositions bivalent*. Woleński gives these axioms in: J. Woleński, *Filozoficzna szkoła lwowsko–warszawska*, pp. 100–101.

[2] J. Kotas, A. Pieczkowski, *Scientific work of Stanisław Jaśkowski*, p. 9.

[3] Jaśkowski gave the results of his research on intuitionistic logic in: S. Jaśkowski, *Recherches sur le système de la logique intuitioniste*. Our presentation is based on Zygmunt Zawirski's article, *Geneza i rozwój logiki intuicjonistycznej*.

and in intuitionistic logic:

p	$\sim p$
1	0
0	1
½	0

The matrix for negation is as follows in classical logic:

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

and in intuitionistic logic:

p	q	$p \rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1
0	½	1
½	½	1
1	½	½
½	0	0
½	1	1

The Heyting's system has some faults. It turned out that the truth tables are inadequate, i.e. they are satisfied not only by Heyting's axioms, but also by formulas which do not follow from these axioms, and so do not belong to the system. On the other hand, it turned out, from Gödel's studies, that an adequate matrix for Hayting's logic must be an infinitely many–valued matrix. Jaśkowski's research allowed this infinitive sequence of intuitionistic logic systems to be better understood.

Matrices have their main trunk and branches. Each branch has an infinitive series of new matrices. The main matrices make the following scheme[1].

The distinguished value (truth) is denoted by "1". The not–distinguished values in successive matrices grow by one value each time.

Let us take function α. When its argument has value 1, the value of this function is a new value, which was not present in the previous matrix. When value of this function is a not–distinguished value in the previous matrix, value α is equal to this not–distinguished value.

[1] Z. Zawirski, *Geneza i rozwój logiki intuicjonistycznej*, p. 206.

In a one–valued logic there is only 1 value which is a distinguished value. A new value, not present in single–value logic, is introduced in two–valued logic by function α, then $\alpha(1) = 0$. There was not distinguished values in the one–valued logic, so we do not use the second property of this function. Now when we pass from the two–valued logic to a three–valued logic, we obtain:

$\alpha(1) = 2$, $\alpha(0) = 0$

When we pass to four–valued logic, we obtain:

$\alpha(1) = 3$, $\alpha(0) = 0$, $\alpha(2) = 2$

In this way Jaśkowski provided a general method of building a matrix when we know the structure of the previous matrix. This method can be presented as in the following table[1]:

$\rightarrow$	1	$\alpha(y)$
1	$1 \rightarrow_M 1$	$\alpha(1 \rightarrow_M y)$
$\alpha(x)$	$x \rightarrow_M 1$	$x \rightarrow_M y$

The matrices for implication in one–valued logic have the form:

$\rightarrow$	1	$\alpha(1)$
1	$1 \rightarrow 1$	$\alpha(1 \rightarrow 1)$
$\alpha(1)$	$1 \rightarrow 1$	$1 \rightarrow 1$

We obtain from it:

$\rightarrow$	1	$\alpha(1)$
1	1	$\alpha(1)$
$\alpha(1)$	1	1

The matrix for the two valued–logic:

$\rightarrow$	1	$\alpha(0)$
1	1	$\alpha(0)$
$\alpha(1)$	1	1

For the three–valued logic, the matrix will have following form:

$\rightarrow$	1	$\alpha(1)$	$\alpha(0)$
1	$1 \rightarrow 1$	$\alpha(1 \rightarrow 2)$	$\alpha(1 \rightarrow 0)$
$\alpha(1)$	$1 \rightarrow 1$	$1 \rightarrow 1$	$1 \rightarrow 0$
$\alpha(0)$	$0 \rightarrow 1$	$0 \rightarrow 1$	$0 \rightarrow 0$

etc.

[1] Z. Zawirski, *Geneza i rozwój logiki intuicjonistycznej*, pp. 207–208, J. Kotas, A Pieczkowski, *Scientific works of Stanisław Jaśkowski.*

Jaśkowski gives a theorem, which says that the set of Heyting's logic theorems is identical with the set of theorems that is fulfilled in an indefinite multi–valued matrix[1]. The sequence of Jaśkowski's matrices is characteristic for intuitionistic propositional calculus.

Jaśkowski gave also axioms for the intuitionistic sentential calculus. He reduced Heyting's number of axioms from twelve to ten. They were presented by Zawirski in 1946[2].

5. The causal logic

5. 1. The expression "if...then..." has different meanings in language. Sometimes it is used in following manner: "if *p*, then from this reason *q*". This is the conditional or causal use of the function "if...then...". Classical calculus does not express a causal relation. For that reason, philosophers and logicians create non–classical systems of logic. Some ideas were formulated in the fifties by Stanisław Jaśkowski. In fact, he laid the foundations of causal logic. This causal logic consists of three types of causal implications: factorial, efficient and definitive.

Jaśkowski's causal logic is based on the sentential calculus with dependent variables. The notion of dependent variable is taken from mathematics. In mathematics the function $f(x)$ is represented by letter y, formally: $y = f(x)$. The variable x is independent. As the value of y depends on the value of x, y is called the dependent variable[3].

According to Jaśkowski, the dependent sentential variables represent sentences, whose truth or falsehood depends on certain arguments. The value of sentences depend on accidental events such as the results of random choice, decision, atmospheric conditions, unforeseen circumstances etc. In matter of facts sentential variables represent sentential functions. The values of sentences depend on arguments of functions, for example p depends on value x in function $P(x)$. In another words, we can to present propositions with the used sentential functions in the form $P(x_1...x_n)$[4]. Replacing name of an object by x_1 ... x_n, we obtain sentence from sentential function. Let P denotes the property of being musician, x denote the set of humans. If we substitute the variable x by the constant name *Krzysztof Penderecki* in the sentential function $P(x)$ we obtain the true sentence *Krzysztof Penderecki is a musician*, but if we substitute x by the name *Tadeusz Różewicz*, we get false sentence *Tadeusz Różewicz is a musician*, and similarly with the predicates of many arguments. The dependent sentential variables denote sentences the truth or the falsehood, of which depends on some arguments[5]. These arguments represent some things and their logical type is undefined.

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[1] Jaśkowski's article contains only frame of this theorem. See S. Jaśkowski, *Recherches sur le système de la logique intuitioniste*. There is a complete reconstruction of this prove in: S. J. Surma, *Jaśkowski's matrix criterion for the intuitionistic propositional calculus*, pp. 87–121.

[2] Z. Zawirski, *Geneza i rozwój logiki intuicjonistycznej*, pp. 219–220.

[3] S. Jaśkowski, *Sur les variables propositionnelles dependantes*, p. 17.

[4] S. Jaśkowski, *Sur les variables propositionnelles dependantes*, p. 18.

[5] S. Jaśkowski, *On the modal and causal functions in symbolic logic*, p. 72.

The calculus of the dependent sentential variable Q consists of variables of two kinds: p, q, r ... p_1, p_2,... p_n, the dependent sentential variables; x_1, x_2,..., x_n, the independent variables. We assume a function symbols of the sentential calculus and the quantifiers bounding the independent variables. Let α be the variable which containing the expression with the independent variables, for example $P(x)$. Let CP denote the calculus of predicates. We call a transcription $T(\alpha)$ of expression α into calculus of predicates CP, the expression obtained from α by means of the substitution: $p/P(x_1...x_n)$, $q/Q(x_1...x_n)$ etc. Jaśkowski defined system Q as follows:

D 1. $\alpha \in Q$ if and only if $T(\alpha) \in CP$

The calculus Q is the base for the calculus of factors QF and the calculus of chronological succession of factors $QChF$. The first of them, QF, is grounded on the fact that not every function depends on all arguments. The sentence which is obtained from the sentential function as a result of substituting constants for variables is true for ones and false for another. It is necessary to distinguish between relevant arguments on which the true of the sentence really depends, and irrelevant ones. Jaśkowski called the relevant arguments *factors of sentence*.

5. 2. System QF consists of only dependent sentential variables, function symbols of the sentential calculus. Other than this, there are other logical constants: the general quantifier of factors: $\forall_f$, which is read *for all values of factors*, and a particular quantifier of factors $\exists_f$, which is read *for some values of factors*. If α and β are also formulas belonging to QF, expressions $[\forall_f\ \alpha]\beta$ and $[\exists_f\ \alpha]\beta$ are formulas of QF. The first of them is read: "for all values of the factors of α, it occurs that β", and the second: "for some values of the factors of α, it occurs that β"[1].

Jaśkowski puts abbreviations standing for certain expressions of the system Q. Let X^n denotes sequence of variables x_k ... x_n. S_1 ... S_k are non–empty subsequences of X^n, and $k = 2^n - 1$. Jaśkowski defined a meaning of the term *the factor of sentence* as follows (sign "=:" means *stands for* and "ABR" means *abbreviation*):

ABR1. $f^1(x_i, \alpha) =: \sim \forall x_1 ... \forall x_n (\alpha \rightarrow \forall x_i\ \alpha)$

The expression $f^1(x_i, \alpha)$ says that x_i is the factor of α. More precisely, in each algebraic structure in which $f^1(x_i, \alpha)$ is satisfied, x_i is the factor of α. If $\sim f^1(x_i, \alpha) \in CP$, then x_i is not the factor of α. If the expression $\alpha \vee \sim\alpha$ belongs to CP, it has no factors, and the contradictory expressions like $\alpha \wedge \sim\alpha$, has no factors[2]. The right side of the definition say that not every variable is a factor of α.

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[1] S. Jaśkowski, *On the modal and causal functions in symbolic logic*, p. 78.

[2] A. Pieczkowski, *Causal implications of Jaśkowski*, p. 170.

Let us denote all variables x belonging to S by letter $y_i \dots y_j$.

ABR 2. $f^{\mathrm{I}}(S_i, \alpha) =: . f^{\mathrm{I}}(y_1, \alpha) \wedge \dots \wedge f^{\mathrm{I}}(y_j, \alpha)$

The meaning of ABR 2 is as follows: each variable of S_i is a factor of α. We read in this way general quantifier $\forall_{\mathrm{f}}\, \alpha$.

ABR 3. $[\forall^{n}{}_{f}\alpha]\beta =: \beta \wedge [f^{\mathrm{I}}\,(S_1, \alpha) \to (\forall S_1)\beta] \wedge \dots \wedge [f^{\mathrm{I}}\,(S_k, \alpha) \to (\forall S_k)\,\beta]$

The expression β is true for all values of the variables of sequence $x_k \dots x_n$, which are factors of α. This means that for each value of the arguments $y_1 \dots y_k$ belonging to S_k, where $y_1 \dots y_k$ are factors of α, each argument of the sequence of S_k is also factor of β.

The formula belonging to CP which can be written in the form $[\forall^{n}{}_{f}\, T(\alpha)]\mathrm{T}(\beta)$ is called the transcription of $[\forall_f \alpha]\beta$ with respect to variables $x_1 \dots x_n$[1]. In another words, for $\alpha \in QF$, replacing in α each sign $\forall_f$ by $\forall^{n}{}_{f}$ and $\exists_f$ by $\forall^{n}{}_{f}$ is called transcription of α with respect to the variables $x_1 \dots x_n$ and denoted by $T^{n}{}_{f}\,(\alpha)$[2]. Function T is an interpretation of QF in Q.

Now, we can define "the theorem of system QF":

D2. $\alpha \in QF$ if and only if $T^{n}{}_{f}\,(\alpha) \in Q$

The definition of factorial implication has the following form:

D3. $\alpha \to_f \beta =_{\mathrm{def}} [\forall^{n}{}_{f}\alpha]\,(\alpha \to \beta)$

The meaning of this definition is as follows: whatever is the possible course of those events which may have influence on α, in view of the real course of other events α implies β[3].

5. 3. In system QF we do not assume that events are chronologically ordered. The next calculus, called sentential calculus of chronological succession of factors ($QCSF$), assumes that variables $x_1 \dots x_n$ represent chronologically ordered objects. According to Jaśkowski, sentence p relates objects taking place in a space and occurring at a moment of time. It may happened that the truth of the sentence is possible only up to a certain moment t_1 earlier than t_2. Jaśkowski assumes that variable x represents possible courses of events in a given time. The variables constitute the chronologically ordered sequence $x_1 \dots x_n$. Over the course of time, the earlier arguments take on constant values. In this way, the set of possible factors of each sentence decreases. The variables $x_1 \dots x_{k-1}$ receive values $a_1 \dots a_{k-1}$ and $x_k \dots x_n$ do not have constant values. The set of possible factors is reduced to the sequence $x_k \dots x_n$.

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[1] A. Pieczkowski, *The axiomatic system of the factorial implication*, p. 43.

[2] S. Jaśkowski, *On the modal and causal functions in symbolic logic*, p. 78.

[3] S. Jaśkowski, *On the modal and causal functions in symbolic logic*, p. 89.

The set of meaningful expressions of *QCSF* will contain signs accepted in *QF* and quantifier symbols: the definitive quantifiers $\forall_d$ $\exists_d$ and the efficient quantifiers $\forall_e$ $\exists_e$.

The truth of sentence p for values a_1 ... a_k of the variables x_1, ... x_k does not depend on values a_{k+1} ... a_n of variables x_{k+1} ... x_n. But if p does not depend on the values a_1 ... a_{k-1} of variables x_1 ... x_{k-1}, then the truth of p depends on the arguments x_k ... x_n. In this case, the argument x_k is the efficient factor for the proposition p with respect to the sequence of values a_1 ... a_k of variables x_1 ... x_k. The variables x_k ... x_n constitute the definitive set of arguments with respect to p and values of a_1 ... a_k of variables x_1 ... x_k. According to Jaśkowski, the efficient factor is the last event in the given sequence of events. The value a_k is a cause of p. From this, if p is necessary, i.e. the value of p depends on each variable of sequence x_1...x_n, and then there is not any factor by which depend p. If p is impossible, p does not any efficient factor either[1].

We assume that α, β... belong to *QCSF*, then $[\forall_e\alpha]\beta$ and $[\exists_e\alpha]\beta$ belong to *QCSF*. The formula $[\forall_e\alpha]\beta$ means that for each value of the efficient factor of α, it occurs that β; formula $[\exists_e\alpha]\beta$ means that for some value of the efficient factor of α it occurs that β. We can obtain the exact meaning of those expressions in language of the calculus of the dependent sentences *Q*. Jaśkowski gives the following abbreviations:

ABR 4. $C^n{}_k(\alpha) =: \forall x_k \ldots \forall x_n\ \alpha \vee \forall x_k \ldots \forall x_n \sim\alpha$

The expression "$C^n{}_k(\alpha)$" means that α does not depend upon x_k ...x_n for given values a_1 ... a_{k-1} of x_1 ... x_{k-1}. A disjunction indicates that for each value of variables x_k ... x_n, the formula α is satisfied or for any value of variables x_k ... x_n the formula α is not satisfied. The sentence α has no efficient factors among the variables x_k ...x_n. The predicate C means *does not depend on*, thus $\sim C$ means *depends on*, and the predicate E, *is the efficient factor*.

ABR 5. 1. $E^n{}_k(\alpha) =: C^n{}_1(\alpha)$, for $k = 0$

Expression α is necessary or impossible, i.e. there is no efficient factor for α. In another words, α has 0 efficient factors.

ABR 5. 2. $E^n{}_k(\alpha) =: \sim C^n{}_k(\alpha) \wedge C^n{}_{k+1}(\alpha)$, for $k = 1$...n–1

$E^n{}_k(\alpha)$ is read as "x_k is the efficient factor of E" that means "the truth of α does not depend on variables x_{k+1}...x_n, but depends on x_k." We get

$\sim C^n{}_k(\alpha) =: \sim(\forall x_k \ldots \forall x_n\ \alpha \vee \forall x_k \ldots \forall x_n \sim\alpha)$

from ABR 4. Next we infer

$\sim C^n{}_k(\alpha) =: \sim(\forall x_k \ldots \forall x_n\ \alpha) \wedge \sim(\forall x_k \ldots \forall x_n \sim\alpha)$,

and then $\sim C^n{}_k(\alpha) =: \exists x_k \ldots \exists x_n \sim\alpha \wedge \exists x_k \ldots \exists x_n\ \alpha$. This means that α is true for certain arguments of the sequence of x_k ... x_n and α is not true for certain

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[1] S. Jaśkowski, *On the modal and causal functions in symbolic logic*, p. 81.

arguments of sequence of $x_k \dots x_n$. Because $C^n{}_{k+1}(\alpha)$, then

$\forall x_{k+1} \dots \forall x_n\ \alpha \vee \forall x_{k+1} \dots \forall x_n \sim\alpha$,

we get

$\forall x_{k+1} \dots \forall x_n\ \alpha \vee \sim\exists x_{k+1} \dots \sim\exists x_n\ \alpha$. If sequence $x_{k+1} \dots x_n$ does not include the efficient factor, and the sequence $x_k \dots x_n$ includes this efficient factor, then x_k is the efficient factor of α.

ABR 5. 3. $E^n{}_k(\alpha) =: C^n{}_k(\alpha)$, for $k = n$

In this case x_n is the efficient factor of α.

ABR 6. $[\forall^n{}_e\alpha]\beta =: (E^n{}_0(\alpha) \wedge \beta) \vee (E^n{}_1(\alpha) \wedge \forall x_1\ \beta) \vee (E^n{}_2(\alpha) \wedge \forall x_2\ \beta) \vee \vee \dots \vee (E^n{}_n(\alpha) \wedge \forall x_n\ \beta)$

$[\forall^n{}_e\alpha]\beta$ is the abbreviation for the disjunction having $n+1$ elements. If only one element of this disjunction is true, then α has the efficient factor. ABR 6 is read in the following manner: for each value of the efficient factor of α, it occurs that β.

Pieczkowski gives an abbreviation that efficient implication is easier for understanding:[1]

ABR 7. $\alpha \rightarrow^n{}_e \beta =: [\forall^n{}_e\alpha](\alpha \rightarrow \beta)$

For every value of the efficient factor of α, the implication $\alpha \rightarrow \beta$ occurs. The efficient factor is the factor at the moment of time. The moment t at which it has been decided that α or $\sim\alpha$ is true, has been decided that $\alpha \rightarrow \beta$ is true. The expressions belonging to Q are analogous to the expressions belonging to *QCSF*. The atomic formulae of *QCSF* (dependent sentential variables: $p, q, r\dots$) correspond to the atomic formulae of *CP* (sentential function $P(x_1 \dots x_n)$, $Q(x_1 \dots x_n)\dots$). The compounded formula $\alpha \rightarrow \beta \in Q$ corresponds to $\alpha' \rightarrow \beta' \in QCSF$, and $\sim\alpha \in Q$ corresponds to $\sim\alpha \in QCSF$, and $[\forall^n{}_e\alpha]\beta \in Q$ corresponds to $[\forall_e\alpha']\beta' \in QCSF$[2].

Jaśkowski introduce in *QCSF* the definitive implication. He puts the general definitive quantifier $[\forall^n{}_d\alpha]\beta$ in the predicate calculus:

ABR 8. $[\forall^n{}_d\alpha]\beta =: [C^n{}_2(\alpha) \rightarrow \forall x_1 \dots \forall x_n\beta] \wedge [C^n{}_3(\alpha) \rightarrow \forall x_2 \dots \forall x_n\ \beta] \wedge \wedge \dots \wedge [C^n{}_n(\alpha) \rightarrow \forall x_{n-1}, \forall x_n\ \beta] \wedge \forall x_n\ \beta$

The expression $[\forall^n{}_d\alpha]\beta$ is a short for the n–ary conjunction and it means that for all values of the variables belonging to the definitive set $x_k \dots x_n$ in regard to α, it occurs that β. If x_k is an efficient factor of α, the sequence $x_k \dots x_n$ is called the definitive set of arguments with regard to α and the values of

[1] A. Pieczkowski, *On the definitive implication*, p. 171.

[2] A. Pieczkowski, *The efficient implications*, p. 8.

variables $x_1 \dots x_k$. If α has no any efficient factor, all variables $x_1 \dots x_n$ constitute the definitive set of α[1].

Analogically to the case of sign $\rightarrow''_e$ we get abbreviation for the sign $\rightarrow''_d$:

ABR 9. $\alpha \rightarrow''_d \beta =: [\forall''_d \alpha](\alpha \rightarrow \beta)$

Generally, we can say that a symbol $T''_{QCSF}(\alpha)$ is the symbol of transcription of the formula α belonging to the system $QCSF$ into calculus of predicates with respect to variables $x_1 \dots x_n$[2].

D 4. $\alpha \in QCSF$ if and only if $T''_{QCSF}(\alpha) \in Q$

Our consideration allows us to formulate definition of efficient implication:

D 5. $\alpha \rightarrow_e \beta =_{\text{def}} [\forall_e \alpha]\ (\alpha \rightarrow \beta)$

Jaśkowski explains the meaning of D5 as follows: in view of the real course of those events which are preceding and of those which are succeeding, the moment t at which it has been decided that α or $\sim\alpha$, whatever is the possible course of those events which are simultaneous with t, should α occur, β would also occur[3].

D 6. $\alpha \rightarrow_d \beta =_{\text{def}} [\forall_d \alpha]\ (\alpha \rightarrow \beta)$

We can give two interpretations of the definitive implication: one when α is true and one when α is false. (1) There is the real course of events, which is previous to moment t. If at moment t it was decided that α, it was also certain that β whatever the possible courses of events coming after t. (2) There is the real course of events previous to the moment t. If at moment t it was decided that $\sim\alpha$, then whatever the possible courses of events simultaneous with t or events succeeding t, if α occurred, β would also occur[4].

The logical questions concern the logical structure of sentences that express causal relations. Jaśkowski is interested in a formalization of causal functions that will allow him to express the conditions necessary for an occurring effect. He knows that a cause as sufficient condition encounters many difficulties. If we accept that the sufficient condition consists of many necessary conditions, we never find all them. Jaśkowski wrote that the given definition does not exhaust the problem of the formalization of causal functions and that the functions defined are insufficient in some cases[5].

[1] A. Pieczkowski, *On the definitive implication*, p. 102.

[2] A. Pieczkowski, *The efficient implications*, p. 9 & p. 10.

[3] S. Jaśkowski, *On the modal and causal functions in symbolic logic*, p. 89.

[4] S. Jaśkowski, *On the modal and causal functions in symbolic logic*, p. 89.

[5] S. Jaśkowski, *On the modal and causal functions in symbolic logic*, p. 91.

6. The first system of paraconsistent logic: discursive logic

The idea of paraconsistent logic is connected to doubts concerning the principle of non–contradiction. In this sense it has similar origins to Łukasiewicz's three–valued logic. According to Jaśkowski, Aristotle's view on the general validity of a principle of non–contradiction was not the only view. Heraclitus, for example, was among its opponent. In demonstrating the need to undertake study of the principle of non–contradiction, Jaśkowski drew also on later, nineteenth–century tradition. Hegel and Marx took up the dialectical ideas of antiquity. But history is not the only nor the principle ground on which the undertaking is to be justified. Jaśkowski also raises the normal practice of language use. In everyday language the principle of non–contradiction is very often ignored. Human knowledge, or one's *Weltanschauung*, contains mutually contradictory theses. This is due above all to the instability of natural language's expressions, in which we express our convictions[1].

The principle of non–contradiction accepted in logic does not really differ significantly from that formulated by Aristotle. However Jaśkowski holds that the Aristotelian principle is incomplete, and for this reason natural language can *break* it. It should be completed by saying that two contradictory statements are simultaneously true regarding one language. This is to prevent the use of expressions which with the passage of time, or in the mouths of different people, do not have the same meaning. An unstable name can lead to contradiction because it can say of one and the same object both "*a* is *A*" and immediately afterwards "*a* is not *A*", depending on the meaning intended each time. Another reason for the construction of a discursive logic is the appearance of hypotheses in science that do not agree with each other, yet which are supposed to explain the same phenomenon.

We say a deductive system is contradictory if among its theses are found two mutually contradictory theses, T and $\sim T$. If a contradictory system is based on two–valued logic, then by the implicational rule of overflowing, we can obtain in it as a thesis any expression W which is meaningful in this system.

Jaśkowski based the system of discursive logic on the modal calculus M5 (in his terms M_2). This modal calculus, in turn, like the causal logic, he connected with the dependent sentential calculus. In relation to classical calculi, modal logic also accepts the modal operators *it is possible that* and *it is necessary that*. The truth of the sentence α depends on certain extra–linguistic factors. The sentence α can be treated as a function taking on the values of true and false depending on the values of the variables representing chance events.

Jaśkowski understands the expression Kp, "it is necessary that p" as saying *in all possible courses of events event p is true*. Having the concept of necessity he introduces a second modal function, that of possibility. Mp means "it is possible that p". "It is possible, that p" can be defined "it is not necessary that p". Formally:

[1] S. Jaśkowski, *Rachunek zdań dla systemów dedukcyjnych sprzecznych*, p. 59.

$Mp = \sim K \sim p$

Accepting the quantifier interpretation of modal operators we will say that the expression *it is possible that* corresponds to the existential qualifier, here read as *for certain courses of events.* in this interpretation the expression *it is necessary that* corresponds to the universal quantifier, read *for every course of events* [1].

Let us suppose that we introduce to one deductive system theses that do not satisfy condition of not containing expressions with unstable meanings. It is enough to introduce consequences from a few mutually inconsistent hypotheses the nature of theses will be changed – they will not express an uniform view. We will obtain the same results by connecting theses expressed by several discussion participants into a single system. The same holds for our own views when we are not sure that there is not some subtle differences of meaning in expressions occurring in different theses.

Jaśkowski calls a discursive system a system about which we can not say that its theses are mutually consistent. In order to demonstrate the nature of theses in a discursive system, each of them would have to be preceded by the caution *in the opinion of one of the discussion's participants* or *with some acceptable usage of words*. Therefore, introducing of a thesis to the discursive system has different intuitive meaning than recognition, i.e. assertion in a normal system. The discursive assertion includes *implicitly* a reservation of some kind, e.g. one of those just given which finds an equivalent in possibility among the logic functions introduced. Therefore, in the discursive logic, thesis T has sense MT, i.e. it is possible that T. Such a basic reasoning method as the law of detachment fails in discursive logic. If implication is understood as in two–valued logic, then from two theses, where one has the form:

$$\text{(i) } \alpha \rightarrow \beta$$

which in discursive logic says, that "it is possible that if α then β" and the second thesis has the form:

$$\text{(ii) } \alpha$$

i.e. it is possible that α, it does not yet result that it is possible that β, and, therefore, β does not result, as would be required by law of detachment[2].

Jaśkowski introduces discursive implication by means of definition[3]:

$$p \rightarrow^{d} q =_{\mathrm{df}} Mp \rightarrow q$$

[1] S. Jaśkowski, *Rachunek zdań dla systemów dedukcyjnych sprzecznych*, pp. 64–65.

[2] S. Jaśkowski, *Rachunek zdań dla systemów dedukcyjnych sprzecznych*, p. 66.

[3] S. Jaśkowski, *Rachunek zdań dla systemów dedukcyjnych sprzecznych*, p. 67.

read: if it is possible that p, then q. Jaśkowski called his the discursive system based on this implication "D" with subscript "2" D_2.

In the discursive system D_2, q results from two theses: $p \rightarrow^d q$ and p. The law of detachment can be applied therefore to discursive implication, since the formula $M[(Mp \rightarrow q) \rightarrow (Mp \rightarrow Mq)]$ is a thesis in modal calculus S5. The modal interpretation of discursive equivalence can be provided similarly. Jaśkowski proves further that each thesis of classical propositional calculus, that does not include other functors than implication, alternative and equivalence, becomes a thesis of D_2, if "$\rightarrow$" is replaced by "$\rightarrow^d$" and "$\leftrightarrow$" by "$\leftrightarrow^d$". Therefore, both a principle of non–contradiction $\sim(p \wedge \sim p)$ and conjunction rule of overflowing $(p \wedge \sim p) \rightarrow^d q$ are thesis of discursive logic. The rule of overflowing is tightly bound to the whole idea of Jaśkowski's discursive logic. The conjunction rule of overflowing makes any system of discursive logic contradictory. However here rejection of implicative rule of overflowing $p \rightarrow^d (\sim p \rightarrow^d q)$ is of crucial important. It permits the existence of contradictory opinions without causing overfilling of discussion.

The system D_2 can be supplemented by introducing to it a classical functors "$\rightarrow$" and "$\leftrightarrow$". If we will define material implication in the following way:

$$p \rightarrow q = \sim p \vee q$$

we obtain all implication–negation theses with the implicative rule of overflowing. This rule will not read to overflowing of each contradictory system, because we do not have detachment rule for material implication in D_2[1].

The implicational rule of overflowing did not belong to D_2, because the expression $M[Mp \rightarrow (M\sim p \rightarrow q)]$ is rejected in system $\mathbf{M}_2$. To demonstrate this statement let us substitute possible statement but not the necessity statement by p, and statement impossible by q. Then the antecedents Mp and $M\sim p$ are true but the complete statement is false. The rejected D_2 allows contradictory statements to occur in discussion[2]. The contradictory sentences may be said by several people at the same time or by one person at different times. It would make overfilling of system on the base of conjunctive rule of overflowing[3].

7. Conclusion

Jaśkowski applied the achievements of modal and intuitionistic logic in his studies on non–classical logics. He used mainly concept of dependent variable in defining the modal operators and, therefore, also a functions of discursive logic and causal logic. This was probably the first quantifier interpretation of the possibility and necessity operators.

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[1] S. Jaśkowski, *Rachunek zdań dla systemów dedukcyjnych sprzecznych*, p. 70.

[2] S. Jaśkowski, *Rachunek zdań dla systemów dedukcyjnych sprzecznych*, pp. 71–72.

[3] L. Dubikajtis, *Stanisław Jaśkowski*, p. 20. For Woleński, the discursive logic system is the most interesting and most important from the point of view of modern logical studies. J. Woleński, *Filozoficzna szkoła lwowsko–warszawska*, p. 132.

The truth tables for intuitionistic logic developed by Jaśkowski are positive supplement of Gödel's studies. They permit the organization of an indefinite number of intuitionistic calculus logical values within an accurate system.

The causal logic was the first logical system describing causal functions in Poland. Jaśkowski was not interested in unchanging interrelations among objects as in physics, chemistry etc. He would treat the cause as sufficient condition for the effect. The object that is a cause depends on different other objects. Jaśkowski showed the logical structure of sentences expressing the relations of causes and effects of an events.

The discursive logic is a logic which allows for contradiction in discussion. Two persons in discourse may to have inconsistent beliefs. In the science scientists may put inconsistent hypotheses forward and system or theory will not be overfilled.

Jaśkowski was the first logician who developed natural deduction calculus in studies on classical calculus of sentences. Unfortunately, for health reasons, he could not publish results of his studies in 1920s.

The Jaśkowski heritage is not large, but majority of texts, which he wrote, had impact on development of the world logic. They were mainly detailed studies, concerning specific problems and they were always innovative. Both causal and discursive logic are only basic intuitions which were developed in the next years.

Supplement

1. Stanisław Jaśkowski was born in Warsaw on 22nd April 1906 in a landowning family, the son of Feliks Jaśkowski and Kazimiera Dzierżbicka. He left secondary school in Zakopane in 1923 and began studies in mathematics faculty of Warsaw University in the same year. During his studies at Warsaw University he attended Jan Łukasiewicz's lectures on mathematical logic. Leśniewski and Tarski were also his teachers[1]. It seems that Łukasiewicz's person and views played the most important role in Jaśkowski's scientific life. Jaśkowski gained his doctorate in 1932 for a dissertation on natural deduction system. His supervisor was Jan Łukasiewicz. The dissertation was not printed until 1934. The habilitation colloquium at the Jagiellonian University in Cracow was conducted on 1st October and concerned a new definition of real numbers. Zygmunt Zawirski was Jaśkowski's supervisor. The habilitation was confirmed on 7th April 1946 and Jaśkowski obtained the title of associate professor on 7th July 1946 (confirmed on 24thJuly)[2].

In 1939 he served as a volunteer in the defence of Warsaw in the 151st Column of Heavy Trucks. After the September campaign, Jaśkowski moved to his estate in Wolka near Rawa Mazowiecka. In 1942 he was arrested for a week. He was released and stayed in his father's estate in Chociwie to the end of war.

After the war, from 1st April 1945, Jaśkowski conducted commissioned lectures at the newly established University of Łódź. The deputy vice

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[1] L. Dubikajtis, *The Life and Works of Stanisław Jaśkowski*, p. 109.

[2] L. Dubikajtis, *The Life and Works of Stanisław Jaśkowski*, p. 110.

chancellor was Ludwik Kolankowski, to whom was entrusted the organization of a University in Toruń. He proposed to Jaśkowski that he should take the department of mathematical logic. As Toruń University was continuation of Vilnius University, the mathematics faculty structure was the same as in Vilnius. There were three departments of mathematics plus a mathematical logic department. As there were problems in all the departments with finding professors, due to Juliusz Rudnicki's illness and Antoni Zygmund's refusal to return to Poland from the USA, Jaśkowski had to organize the departments and conduct lectures alone. He had to retrain and to undertake lectures on mathematical analysis, set theory, differential geometry and probability theory. There was a real danger in that period that the mathematical faculty would be closed, but this was prevented by the hiring of Jerzy Łoś and Leon Jeśmianowicz[1].

Jaśkowski worked also from 1950 in the National Institute of Mathematics at the Polish Academy of Sciences. He was nominated a full professor in 1957. He was the director of mathematical departments in Toruń University to 1965, co–founder and the first president of Toruń division of the Polish Mathematical Society. He was a councilor in the Provincial People's Council in 1957–1959. In 1952–53 he organized the Department of Mathematics, Physics and Chemistry and he was its dean in 1953–54. In 1956–59 he was a deputy prorector for science, and in 1959–62 rector of the Nicholas Copernicus University in Toruń.

Jaśkowski fell victim to an infectious jaundice. He developed complications after this illness and died on 16th November 1965. He was buried in the Powązki cemetery in Warsaw[2].

2. Jaśkowski was a mathematician by education and although his main achievements belong to mathematical logic, he also studied the foundations of mathematics. His works deal with the notion of number and the foundations of geometry, and concentrated on problems of decidability. Jaśkowski proved the decidability of the elementary additive Boolean algebra and of the elementary theory of Boolean rings. He proved the undecidability of certain classes of theorems in the theory of groups, topology and Boolean algebra. In the field of modal logic Jaśkowski proved the decidability of the S5 Lewis system. Jaśkowski was interested in elementary problems in geometry. His disciple, Dubikajtis, wrote that Jaśkowski

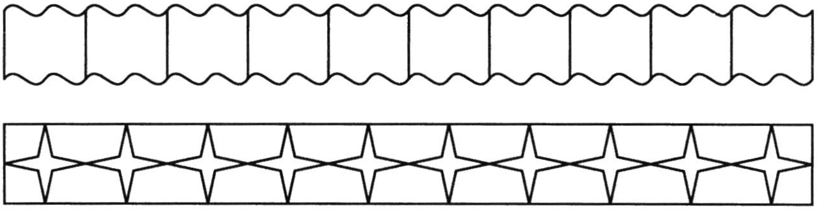
> *was passionately devoted to the problem of eliminating such abstract notions from geometry like the notions of point or of line and wanted to replace them by notions which were more specific, particulary by the notion of solid. Therefore he became interested in Tarski's geometry of solids based on the primitive notion of sphere*[3].

[1] L. Jeśmianowicz, *Stanisław Jaśkowski*, p. 131.

[2] L. Dubikajtis, *Stanisław Jaśkowski*, p. 16.

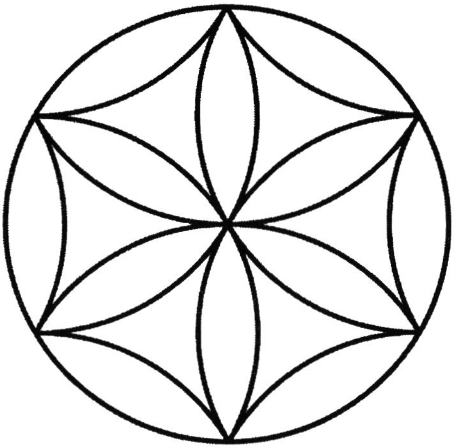
[3] L. Dubikajtis, *The Life and Works of Stanisław Jaśkowski*, p. 111.

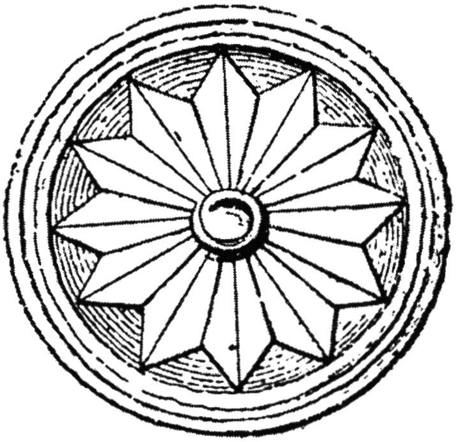
3. Apart from research into mathematical problems he was a great promoter and popularizer of mathematics. His papers on the mathematics of ornament belong in this course of mathematics popularization. Jaśkowski wrote two books on this topic. In these books there were analyses of possibilities of applying mathematics to the description of rosettes, mosaics, architecture, and even to ballet or artistic gymnastic exercises. The idea was to found a method allowing a description of various symmetric patterns. The simplest examples of such patterns are borders consisting of many units of the same figure, named a motif. They motif is displaced in relation to the previous one by a distance defined for the given band. This repetition is called a border rhythm. We observe other kinds of rhythmic repetitions on dishes, rosettes, architectural structures, paintings, poetry, etc[1]. Simple borders have a simple rhythm. Two examples of borders with the simple rhythm are the following:



Jaśkowski tried to demonstrate that symmetries are common above all in nature and that nature is a model for artists. The same can be said about the rotational rhythm. For example, flower petals are arranged in this way. Due to this resemblance, ornaments with rotational rhythm are called the rosettes. A rosette is n–type when the motif is repeated n times and rotation by $1/n$ of round angle takes *the* motif to next position. Columns with motifs repeated rotationally and helically have a different rhythm. There are also architectural compositions where the motif is repeated rhythmically by shifting in three dimensions. They are called spatial lattice ornaments.

[1] S. Jaśkowski, *Matematyka ornamentu*, pp. 9–11.





Above simple rosettes are shown in which we can see that the motif is repeated by rotary motion. We observe less or more complicated motives and different rotary shifts. The property, on which Jaśkowski based the most general classification of ornaments is one geometric nature of rhythm or another. Repetitions of a different type to those we have called rhythm can be observed. For example, the motif subject to rhythmical shifting is often divided into two identical, symmetric parts. Ornament is constructed from various types of repetitions of components, such as is for example, a brick house.

According to Jaśkowski, the regularity of repetition of ornament parts can be described only by mathematical concepts, so we can talk about a mathematical theory of ornament. He is concerned with properties of symmetry and rhythm. Jaśkowski points out that even a primitive drawing, repeated many times in the right way, gives a whole pleasant for the eye. In Jaśkowski's opinion the ornaments of the Egyptian were not made by chance, although no written sources on the theory of ornaments surviv. The principles of ornament painting are connected with the traditions of a given culture. They often have magic and religious meanings and should were to inspire favor with the gods. Drawing skill was one of the ways the development of geometry. Compasses were used for the first time in Elam ornaments, which need quite complex geometric structures.

The main notion of the mathematics of ornament theory is the notion of symmetry. This notion may be generalized on geometric shapes and solids in space. The general notion of symmetry is a notion fundamental to the theory of point transformation[1].

[1] S. Jaśkowski, *Matematyka ornamentu*, pp. 19–20.

According to Jaśkowski, ornaments presenting simple geometrical figures, such as circles, ellipses, polygons, are common and are called geometric ornament. The knowledge of mathematics served once to determine some constant proportions of objects. Now the laws of physics are used in the development of some ornaments. Vegetable and animals motifs and so on are used. The current state of mathematical knowledge permits the development of richer patterns such as crystals, lattice ornaments corresponding to crystalline structures (network of arcs). Jaśkowski indicates that the same contents can be communicated in art in many ways. The knowledge of modern science can help in this; its results can be used in artistic creation. Once, when a builder wanted to emphasize the weight of a protruding structural element, he used a sculpture of a muscular man, supporting the element with effort. The same effect was achieved by the shape of the classical column, which was not an exact cylinder, nor a truncated cone, but was apparently deformed the cross–section of a middle part a little higher than at the base or top. Today we know how an overloaded elastic bar is deformed. If we fix the base of this bar so that it will be directed upwards in an unloaded condition, and then load strongly the bar top, then with a properly selected weight the bar will form a deformed sinusoid curve. The number of sinusoid inflexions depends on the weight supported[1].

Jaśkowski was of the opinion that the dissemination of knowledge of new scientific discoveries, and the raising of the level of mathematical culture, permit new designs taken from scientific research to be added to ornamentation, and previously unknown figures to be used. This is a natural development of ornament, in which mathematical and scientific knowledge finds its culture reflection.

4. Jaśkowski's work for a reform of mathematics syllabus for secondary schools was part of his popularization of mathematics. As he wrote, the point was that learning mathematics should not give a false understanding what mathematics is. He joined in the work modernizing the mathematics syllabus of secondary schools at the turn of 1950s and 1960s. His articles in *Mathematyka* created an atmosphere for discussion on reform and its principles[2]. As a member of the committee for the new mathematics syllabus at the Polish Mathematics Society, he participated in the development of the new syllabus, which was introduced in the 1960s[3].

Jaśkowski highlighted the dramatic situation in mathematics teaching in secondary schools. In his opinion, the content of mathematics teaching in the 1950 did not exceed the state of mathematical knowledge 300 years ago. The scope of material stopped just before the discovery of the integral and differential calculus, so just prior to Leibniz and Newton. He said that if the physics teaching syllabus would be cut in the same way, then not only atomic

[1] S. Jaśkowski, *Matematyka ornamentu*, pp. 91–94.

[2] L. Jeśmianowicz, *Stanisław Jaśkowski*, p. 134.

[3] L. Dubikajtis, *The Life and Works of Stanisław Jaśkowski*, p. 112.

physics, but also Newton's laws of mechanics would have to be removed from it. Jaśkowski held that the programme of teaching in the natural science is updated because new discoveries often prove earlier theories invalid. This situation forces authors of school syllabus to continuously revise the teaching program. In mathematics it is otherwise. New discoveries do not lead to negation of previous theorems. The logical value of any sentence is not changed. Therefore, it is not possible to demand modification of the syllabus in the name of integrity conceived as the obligation to tell the truth[1].

Jaśkowski indicated, that mathematics, as any science, does progresses in specialization. It has ceased to be merely a servicing science, teaching calculating and measuring. It has become a theoretical science. The university syllabus contains above all ever more abstract and general theorems on which the practical applications mathematics are base. Mathematics becomes a theoretical science and ever less an exercise of algorithmic skills, i.e. skills in calculation and measurement. It is likewise in other fields of science, and this process is generally reflected in school syllab. So it is possible to provide pupils with information about nuclear disintegration without requiring them to operate the nuclear reactor. It is different with mathematics[2].

One of the properties of mathematic progress is formalization, i.e. the introduction of symbolic notation instead of description, and, in connection with this, replacement of inference in words by rules for operating with symbols. The memorization of a calculus or detached formalism without care of their proper understanding, are mistakes. Mathematics should develop skills of logical thinking. In Jaśkowski's opinion, modernization does not mean only growth of the syllabus. The essential element is elimination of obsolete material or material less important in the current teaching of mathematics. In his opinion, some elements are even harmful, as they instill in pupils false beliefs about what mathematics is and accustom them to using obsolete calculating methods; the use of compasses, triangle, protractor, is only a means, not an end[3].

According to Jaśkowski, the motto *less but well* should be adopted for teaching mathematics. As minimum requirements in theory lectures the following suffice (1) strict formulation of definition, (2) resignation from competence in using an algorithm, or some proofs, (3) resignation from providing inexact pseudo–proofs, which distort understanding of mathematics[4].

Jaśkowski wanted therefore to bring mathematics closer to other subjects, such as physics or biology, in which lectures are not restricted to laws that can be justified by experiments made in front of pupils. Mathematical proofs may be provided in textbooks and it is not necessary to require their memorizing. The objective of the proof is to convince pupils of truth of some thesis, so the teacher, not the pupil, is obliged to provide it. The scope of the material,

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[1] S. Jaśkowski, *Problem modernizacji materiału programowego z zakresu matematyki* ... , p. 47.

[2] S. Jaśkowski, *Problem modernizacji materiału programowego z zakresu matematyki* ... , p. 48.

[3] S. Jaśkowski, *Problem modernizacji materiału programowego z zakresu matematyki* ... , p. 49.

[4] S. Jaśkowski, *Problem modernizacji materiału programowego z zakresu matematyki* ... , p. 53.

which ought to be obligatory in schools can be divided into two parts: one which increasing systematic material studied in detail (for example, mathematical analysis), the other, encyclopedic material (descriptive review of selected parts of mathematics as a component of general education)[1].

In his thoughts on the construction of the mathematics programme, Jaśkowski held above al to simplicity and transparency, characteristics of modern mathematical systems. The mathematical logic gives relevant tools for that. Ultimately, the teaching of mathematics should be a practical school of logic[2].

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[1] S. Jaśkowski, *Problem modernizacji materiału programowego z zakresu matematyki ...*, p. 54.

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