

Zbigniew Król

Infinity in Mathematics : Development of Platonic Ideas and Methods in Mathematics in Late Antiquity and the Middle Ages = Nieskończoność w matematyce : rozwój idei Platońskich i metod w matematyce w późnej starożytności i średniowieczu

Humanistyka i Przyrodoznawstwo 19, 7-27

2013

Artykuł został opracowany do udostępnienia w internecie przez Muzeum Historii Polski w ramach prac podejmowanych na rzecz zapewnienia otwartego, powszechnego i trwałego dostępu do polskiego dorobku naukowego i kulturalnego. Artykuł jest umieszczony w kolekcji cyfrowej bazhum.muzhp.pl, gromadzącej zawartość polskich czasopism humanistycznych i społecznych.

Tekst jest udostępniony do wykorzystania w ramach dozwolonego użytku.

Zbigniew Król

Politechnika Warszawska

Warsaw University of Technology

INFINITY IN MATHEMATICS: DEVELOPMENT OF PLATONIC IDEAS AND METHODS IN MATHEMATICS IN LATE ANTIQUITY AND THE MIDDLE AGES*

Nieskończoność w matematyce: rozwój idei Platońskich i metod w matematyce w późnej starożytności i średniowieczu

Key words: philosophy of mathematics, history of mathematics, infinity in mathematics, *Elements*, Euclid, Euclidian geometry, medieval mathematics, ancient mathematics.

Słowa kluczowe: filozofia matematyki, historia matematyki, nieskończoność matematyczna, geometria euklidesowa, *Elementy* Euklidesa, matematyka średniowieczna, matematyka starożytna.

A b s t r a c t

The paper is devoted to the reconstruction of some stages of the process leading to the emergence in modern science the concept of infinite “Euclidean” space in geometry of the *Elements* in late antiquity and the Middle Ages. Some historical medieval sources and views concerning Archytas, Cleomedes, Heron, Proclus, Simplicius, Aganis, al-Nayrizi and the Arabs, Boetius, Euclid, Gerard of Cremona, Albertus Magnus *et al.*, are described analyzed and compared. The small changes in the understanding of geometry in the *Elements* during the ages are reconstructed up to the first explicit use of the concept of infinity in geometry by Nicole Oresme.

S t r e s z c z e n i e

Artykuł omawia pewne aspekty procesu historycznego, w wyniku którego w geometrii euklidesowej pojawiło się pojęcie nieskończonej, absolutnej przestrzeni, nieskończonych prostych, płaszczyzn etc. Analizuje się i porównuje źródła historyczne, głównie średniowieczne, dotyczące poglądów i postaw względem nieskończoności w matematyce takich autorów, jak Archytas, Kleomedes, Heron, Proklos, Symplikios, Aganis, al-Nayrizi (i Arabowie), Boecjusz, Euklides, Gerard z Cremony, Albert Wielki i inni. Omawiane są istotne zmiany i innowacje wprowadzane sukcesywnie w ciągu wieków, które doprowadziły do pierwszego świadomego zastosowania pojęcia nieskończoności w twierdzeniach geometrii przez Mikołaja z Oresme.

* The research and the paper are supported financially by the Budget in 2010–2015; the scientific grant nr N N101 058939. (Praca naukowa finansowana ze środków budżetowych na naukę w latach 2010–2015 w ramach projektu badawczego nr N N101 058939.)

In the present paper, I would like to investigate the most important stages of the process leading to the emergence in modern science of a new intuitive (infinite) model for modern mathematics and the calculus in the Middle Ages.

At first glance, it is obvious that modern science of Newton and his followers is based on some infinite notions and infinite mathematics: *absolute space*, *infinite straight lines*, etc. However, they were absent in ancient geometry of the *Elements* [cf. Król 2005]. Therefore, it is necessary to explain how it was possible to exchange the intuitive finite model of ancient mathematics with a different and infinite model. The exchange is not the discovery by a single man. It is the result of a long and complex historical process.

We can demonstrate the existence and peculiarity of the hermeneutical horizon for mathematics in antiquity by showing the results of a long historical case study. We can even make a thought experiment showing the active character of the horizon.

The experiment relies on the possibility of reading the text of the translation of the *Elements* with enough understanding. We can do it ourselves or observe the understanding of the text by a pupil or even a child. It sometimes happens that pupil can state many properties (e.g. “the diameter divides a circle into two equal parts”) without any proof, or even formulate some simple proofs. We can also observe how the famous fifth Euclid postulate is understood: “on the plane one can draw only one parallel straight line to the given one, crossing the given point not contained in the given line”.

We can reconstruct the hermeneutical conditions informing our understanding of Euclidean geometry and we will see that we create and understand the geometry in the determined intuitive model, which is a “part” of the hermeneutical horizon. In our example, the basis is the infinite, rigid, unchangeable, or, in the Newtonian sense, absolute “Euclidean space”, treated as a container or an arena for geometry to play itself out, “the same” in every place and moment of time. When one reads in the *Elements* the word “line”, “surface”, etc., it is understood as “infinite straight line”, “infinite surface” injected in a presupposed infinite space.

Geometrical concepts such as triangles, squares, polyhedra seem the same today as they were in antiquity. Moreover, there is no possibility to understand them as intuitively clear and distinct in any different way.

Of course, we are aware of some changes, such as the possibility of the creation of non-Euclidean geometries. The discovery of non-Euclidean geometry was shocking. It showed that there is the possibility to change something in Euclidean geometry, i.e. in the fifth Euclid postulate. We can imagine some intuitive and clear models, the surface of a sphere for instance, in which every Euclidean “axiom” is true except the fifth postulate and, to obtain this result, it is enough to change the meaning of the term “straight line”.

We think that the discovery of another intuitive model for ancient geometry in which every axiom is understood in a different way than in the modern infinite model maybe also shocking. Certainly it is interesting as a theoretical possibility, but more important is its actual role as the base for mathematical creativity in the times of Plato and Euclid.

What are, then, the main differences between intuitive ancient and modern models for Euclidean geometry? The main difference is the absence of the concept of absolute space and general lack of any infinite notions: infinite surface, infinite straight line, infinite line, asymptote etc. The concept of absolute space does not appear in the *Elements* nor the other infinite notions. Other differences are non-continuity and the non-metrical character of geometrical figures, sections etc. We have to ask once more: how is it possible?

The answer is very complex because the *Elements* is not a work of one person, but contains many different mathematical theories emanating from different times which were assembled and completed by Euclid. So, in the *Elements* we have many different theories which sometimes are simply not directly comparable at all. Moreover, the text of the Euclid's *Elements* did not remain unchanged throughout the ages. It was supplemented many times with additions, commentaries, lemmas, etc. Some parts of the text changed their original meaning in translation. Therefore, it is necessary to investigate the content of the most influential versions of the Euclid's *Elements* and the process of the reception of the *Elements* in the Latin Europe.

Some sources concerning ancient authors

Proclus did not influence medieval mathematics, and there are some other ancient authors, especially Heron or Simplicius, who were more important in the medieval mathematics and philosophy of mathematics. The discussion concerning their views created historically an essential part of the transmission of Euclidean geometry and the *Elements* of Euclid in medieval Europe. As I will explain below, in the discussion and reception of their views, one can find traces of gradual emergence of some infinite concepts in geometry.

Thus the views of Heron and Simplicius are interesting – from the point of view of this paper – only when they are connected with the process of transmission of the *Elements* in Western Europe where the new infinite model emerged. We are also interested in the views of these authors who translated or commented on Euclid's *Elements*.

The main source for ancient commentaries of the *Elements* and certain views of some ancient authors, namely Heron, Simplicius, Boethius, Agapius is the medieval Arabic *Commentary of al-Nayrizi of Euclid's Elements of geometry*.

We know only two survived Arabic manuscripts containing the *Commentary*, i.e. the *Codex Leidensis*, MS OR 399.1 (Ms **L**) and the manuscript *Qom* 6526 (Ms **Q**).

It is not possible to reconstruct the whole text of the *Commentary* from the only Arabic texts of the both aforementioned manuscripts. Nevertheless, we have also some Latin sources containing a translation of the *Commentary*. The most important is the famous translation of Gerard of Cremona (1114–1187). This Latin commentary has, for instance, the text (and some comments) of the definitions Def. I.1–I.3 which is missing from the Arabic sources. The Arabic texts end at the beginning of the book VII and the Latin text of the commentary preserves all ten books of it. In general, the text of the Arabic version of the *Elements* is not translated by Gerard into Latin. He translated (almost) only the commentary.

Four manuscripts of Gerard's *Commentary* are known: Biblioteka Jagiellońska 569, Cracow, f. 1–23 (pp. 7–51), (XIV century; Ms **K**), Biblioteca Nacional 10010, Madrid, f. 13v–36v, 49v–50, (XIII/XIV century; Ms **M**), Bodleian Library Digby 168, Oxford, f. 124–125, (XIV century – abridged version), and Regin. lat. 1268, Vaticano, f. 144–183v, 206r–207v, (XIV century; Ms **V**). The *Manuscript Cracoviensis* was edited by M. Curtze and J. Heiberg in Leipzig in 1899 [cf. Heiberg 1883–9], vol. VIII (M. Curtze, *Anaritii in decem libros priores Elementorum Euclidis commentarii*, pp. 1–252)¹. Tummers has shown that MS **V** is copied from **M** and that **K**, **M** and **V** are based on the other (unknown) common source [cf. LoBello 2003b, p. XXX and Tummers 1994]. S. Brentjes reports that some extracts from the commentary are found also in the manuscript in Mumbai (Mulla Firuz Collection in Mumbai, R I.6, dated by Brentjes on the X century [cf. Brentjes 2001]). The same manuscript preserves also more than some short fragments from the al-Hajjaj II tradition [see Król 2012].

The Qom Manuscript is dated to the XV century. It is necessary to add that the Qom MS has mainly free space left for diagrams and only few of them are inserted into the manuscript [cf. LoBello 2009, p. XIII]. There is no one and new diagram in the part of the book I which is present in Ms **Q** and absent in Ms **L**.

The comparison of the Arabic and Latin version of the commentary leads to a conclusion that they preserve the same text of the *Elements* which is own al-Nayrizi edition supplemented by comments [cf. Brentjes 2001 and LoBello 2003b, p. XXXII]. Brentjes argues, however, that the direct sources of the Latin and Arabic manuscripts are different. No Latin manuscript is an original version prepared by Gerard but they are a result of some later editorial activity. As it concerns interested us at the moment ancient authors, the Latin Gerard's commentary transmits more from Heron's comments than the Arabic sources.

¹ Cf. also the edition of Books I–IV by Tummers 1984 and 1994.

The last part [cf. op. cit., pp. 252–386] of Curtze edition of the text of Ms **K** of al-Nayrizi's commentary contains one more commentary on the book X of the *Elements*, i.e. Muhammad ibn Abd al-Baqi's. This commentary was used in the text for the book X in the manuscript Vat. Reg. lat. 1268 [cf. Busard 1985, p. 135].

Many parts of the translation of the *Commentary of al-Nayrizi of Euclid's Elements of geometry* were copied and used by Albertus Magnus in his *Commentary on book I of Euclid's Elements of geometry* [cf. the critical edition in Tummers 1984 and an English translation in LoBello 2003c]. The main codex with this commentary is the *Codex Vienna*, Dom. 80/45. Moreover, the commentary was the source for Roger Bacon and Campanus of Novara. The latter is more important for us. H. L. L. Busard indicated such places (and manuscripts) explicitly [cf. Busard 1985, pp. 134–135].

Obviously, the above sources are based on some other manuscripts and traditions. Moreover, a commentary usually contains a commented variant of the text of the *Elements*. Summing up the findings of Tummers and Arnzen, one can reconstruct the lines of the development of the al-Nayrizi's tradition. Heron used the original, pre-Theonian Greek text of Euclid's *Elements* and the text of the *Elements* used by Heron, and this material is partially present in al-Nayrizi *Commentary*. Al-Nayrizi uses a pre-Theonite text of the *Elements* which also was used by the Greek commentators on Euclid: Heron, Simplicius, Pappus (and – one can add – also Proclus which is, however, not present at the al-Nayrizi's commentary). On the other hand, al-Nayrizi uses also the Theonite tradition through a Greek post-Theonian text of the *Elements* which was the source for the Arabic al-Hajjaj translation. The same Theonian Greek text was the base of the Ishaq-Thabit Arabic tradition (see below) which itself is the source for the Latin translation by Gerard of Cremona. Ms **L** and Ms **Q** had a common ancestor. Also, both lines of the transmission of al-Nayrizi's *Commentary*, i.e. the Arabic and the Latin, had a common ancestor which is also a common ancestor for their more direct ancestors [cf. Ms φ in Arnzen 2002 p. XXV].

It is necessary to redirect the reader to the more detailed description of the general lines of the transmission of the text of the *Elements* from antiquity to the Middle Ages presented in Król [2012]. This survey will provide some chronology for the emergence of a new intuitive model of geometry.

Infinity in ancient views

The invention and application of the infinite Euclidean space in geometry and in mathematics is possible only if there was an aprioric internal possibility to think of finite ancient Euclidean geometry in some new intuitive frames. Of course,

such a move might be stimulated by the process of a divinization of the space, i.e. from the identification of the space with the infinity of God or with God itself, which was described in great detail by E. Grant [cf. Grant 1982]. However, as we will see, the problem is a purely mathematical one, and, even from the historical point of view, it is independent from the invention of an infinite void space in physics and cosmology.

As E. Grant comments: “[T]he adoption of an infinite space in the seventeenth century resulted primarily from the divinization of space – a process begun in the fourteenth century – and, to a lesser extent, to the needs of physics and cosmology. But it did not arise from any straightforward application of an alleged Euclidean geometric space to the physical world” [Grant 1982, p. 273, footnote 49].

Nevertheless, at the beginning of this section, it is necessary to remark briefly on some ancient views concerning the void physical space and the possibility of the existence of an infinite extramundial empty infinite space. It is a well-known fact that Aristotle denied the existence of an empty place (i.e. a place without any body in it) or a vacuum [cf. for instance his *De caelo* 279a 12–14, 17–18, and the definition of a void in *Physics* 214a 8–19 and in the *De caelo* 279a 14–15]. It was Roger Bacon who first changed the Aristotelian definition of vacuum saying that it is “a space in which there is absolutely no body, nor there is a natural aptitude for receiving any body; but to assume [vacuum] in this way, [is to assume it] beyond the heaven”². Bacon was an inventor of a purely conceptual idea of an empty place beyond the heavens because his definition discerned a place in which there is no possibility of the presence of any body.

Coming back to antiquity, there is one fragment of Archytas of Tarentum preserved in Simplicius’ *Commentary on Aristotle’s Physics* in which one finds the description of the following thought experiment: “If I am at the extremity of the heaven of the fixed stars, can I stretched outwards my hand or staff? It is absurd to suppose that I could not; and if I can, what is outside must be either body or space. We may then in the same way get to the outside of that again, and so on; and if there is always a new place to which the staff may be held out, this clearly involves extension without limit”³.

Although this Archytas’ argument was not known in the Middle Ages [cf. Grant 1982, p. 106], there was known [in the Latin translation by Wilhem of Moerbeke, 1271] the fragment of Simplicius’ *Commentary on De caelo* in which almost the same argument is ascribed to the Stoics [cf. Grant 1982, p. 106–107]. The Stoics, in general, inclined to Aristotle’s physics and they

² Cf. Roger Bacon *Physica*, Book IV, p. 108 in Bacon 1928. I quote the English translation by E. Grant 1982, p. 106.

³ I quote the translation by F. M. Cornford in Cornford 1936, p. 233. It is the translation of the fragment 30 of Eudemos who quoted Simplicius’ *Commentary*.

accepted his finitism. However, we know also the hypothetical reasoning of Cleomedes in which the infinity of the space surrounding the spherical world is argued from the acceptance of the supposition that such a surrounders does exist. Such a vacuum must be infinite because if it is not, it must be delimited by a body. However, there is no body outside the world. Therefore the vacuum, if it exists, must be infinite⁴.

A variant of the above Cleomedes' argument, however, that was not very influential in Latin Europe [cf. Grant 1982, p. 322, footnote 12], was known to the Arabs. Al-Ghazali argued that omnipotent God could create a world bigger than the existing world by one cubit, next by two, four etc. cubits. "Now we affirm that this amounts to admitting behind the world a spatial extension which has measure and quantity, as a thing which is bigger by two or three cubits than another occupies a space bigger by two or three cubits, and by reason of this there is behind the world a quantity which demands a substratum and this is a body or empty space. Therefore, there is behind the world empty or occupied space"⁵.

However, as E. Grant writes: "The infinite space that surrounded the world was the product of cosmological and physical controversy and had nothing to do with any alleged application of Euclidean geometric space to the physical world [...]. From the earliest beginnings, associated with the name of Archytas of Tarentum, all the way to the Scientific Revolution of the seventeenth century, those who fashioned the concept of a dimensional, infinite space paid no homage to Euclid. When Pierre Gassendi argued in behalf of a three-dimensional void space, his supportive appeal to the ancients did not include Euclid but rather Epicurus and Nemesius" [cf. Grant 1982, pp. 107–106].

It is now possible to recover the information about the development of Platonic methods in ancient and medieval mathematics based on the *mathematical* sources presented in the paper [Król 2012]. To this methods belongs the use of some infinite objects in Euclidean geometry such as infinite lines, surfaces and space. To perform such a reconstruction, it is necessary to find the historical limits of ancient strict finitism. As it is in the case of Isaac Newton who thinks of Euclidean geometry in a new infinite model, it is possible to find when the commentators, editors and translators of the *Elements* as well as mathematicians started to think of *some parts*, e.g. of some theorems, of Euclid's geometry with the use of infinite objects. First of all, we will see how the views of some ancient authors are presented in the medieval sources.

⁴ Cf. Ziegler 1891, pp. 14, 16; Czwalina 1927, pp. 5–6 and Grant 1982, p. 107.

⁵ Cf. Grant 1982, p. 322, footnote 12. The argument – preserved in Averroes' *Tahafut al-Tahafut* (i.e. *The Incoherence of the Incoherence*) – is rejected even by al-Ghazali himself as based on imagination only. Cf. also an English translation of the *Tahafut* in Bergh 1954.

Some ancient authors, Heron and Simplicius in the light of medieval commentaries

Obviously, we shall focus on the ancient views concerning some infinite objects and infinity in mathematics as well as some remarks concerning the role of the highest principle(s), the One (and the Dyad)⁶.

The most important in this are the comments of Simplicius. al-Nayrizi quotes some Simplicius' passages concerning the highest principles at the beginning of the Latin text of his commentary (there is no counterpart of this section in the Arabic texts). Simplicius discerns clearly *one*, *unity* and *point*. *Unity* is the principle of discrete magnitudes, *point* that of continuous (spatial, geometrical) magnitudes. “[H]e defined [a point – Z.K.]⁷ by negating that it is the cause of dimensions, and it is necessary that the cause be nearer to not being divided than that which has been caused because it is nearer to *one*, which is the cause of the whole⁸. [...] It does not have dimension, nor is divided, and is the cause of that which does have dimension, and is divided. Wherefore this definition is not appropriate to unity [*neque omnino sit unius*; cf. Heiberg-Curtze 3.5], in that it is not the cause of that, having dimension, which is divided, nor it is altogether of one and the same genus with those [things] that have dimensions” [cf. the translation in LoBello 2003b, pp. 15, 16].

Al-Nayrizi himself completes the Simplicius' comments on *unity*: “[...] the continuous and the discrete are differentiated in position; therefore, the end of motion and an instant [of time] will be nearer to a point on account of the fellowship which is between them because of the continuity, which is not in unity. I, however, say that unity is a thing lacking parts and position and that it is the principle of discrete quantity” [cf. LoBello 2003b, p. 17].

The same way of thinking about some *principles* in mathematics is present in Heron's fragment used by Gerard of Cremona, as well as by Albertus Magnus in his *Commentary on the first book of the Euclid's Elements of geometry*: “A point is the undivided principle of all magnitudes [cf. LoBello 2003b, p. 16 and LoBello 2003c, p. 6]⁹.

We should note add that there is a change in the intuitive foundations of mathematics concerning the creation of mathematical entities from their principles. One can see how *motion* (translations, superpositions, incrisings of sides

⁶ The scenery is neo-platonic, i.e. the Dyad (multiplicity) is a secondary and related to the One principle.

⁷ Cf. the relevant definition: “A point is that which has *no* parts”; Heiberg-Curtze 1.4.

⁸ “[...] magis propinqua uni, qui est causa totius”, cf. Heiberg-Curtze 2.25.

⁹ Cf. also Heiberg-Curtze 3.29-30: “punctus est principium omnium quantitatum indivisum”.

of figures, etc.) is entering gradually into geometry and how this is a new element in geometry in comparison to Plato's "static" way of thinking. Simplicius, al-Nayrizi, Albertus Magnus speak about the movements of a point, a line, a circle, a surface, a body. A line is a principle of a surface *because* when it is moved in the second dimension, it produces a surface, etc. The motion is predominant in Albertus Magnus' *Commentary on the first book of the Euclid's Elements of geometry*. In the last commentary also the concept of *space* is used, and Albert even speaks about a point as if it was a part of space: "Motion, however, is not continuous except from the space over which it occurs, and time gets its continuity from motion, and the being of motion and of time is continuous from space, and a bit of motion and an instant of time are indivisible from the indivisible element of space, which is the point" [see LoBello 2003c, pp. 4–5].

Simplicius commenting on the Euclid's definition of straight line, adds: "[...] for he only defined the finite line in this definition" [cf. the Arabic Ms Q, LoBello's translation in LoBello 2003a, p. 1]. It may suggest that there are also infinite lines. The relevant Latin fragment concerns Definition I.4. of the *Elements*: "[...] for he did not define in this place anything but a finite line" [cf. the translation in LoBello 2003b, p. 18]. Moreover, the Ms Q speaks also about "[...] those [lines – Z.K.] whose length is infinite" (in the same comment), [cf. LoBello 2003c, p. 2]. The counterpart of this fragment is the Latin phrase of Gerard: "and others [lines – Z.K.] infinite" [cf. LoBello 2003b, p. 19]; "et alie infinite" [cf. the Heiberg-Curtze edition, 8.15]. The same situation is with the comment concerning Def. I.6 of the extremities of a surface, i.e. lines. The Leiden manuscript (Simplicius) has: "Euclid here did not speak except about a finite surface. Concerning the infinite and round [rotunda] surface, indeed, he said nothing" [cf. LoBello 2003b, p. 22]. This explains the situation: the examples of infinite (*apeiron*, i.e. indefinite) lines and surfaces are circles and (the surfaces of) spheres because they have no ending points as the straight lines. Cf. also some previous comments concerning two classifications of lines preserved in Proclus' *Commentary on the first book of Euclid's Elements* [see Proclus 1992, the first classification pp. 111, 1–9, the second: 111, 9–20, 112, 16–18, Friedlein; cf. also Heath 1908, vol. I, pp. 160–162].

Moreover, the above explanation is congruent to the other fragments from Simplicius preserved in the commentary of al-Nayrizi which explicitly deny the existence of (actual) infinite objects. "Euclid did not say that every line is made finite by points [sit finita punctis]. It is, nevertheless, impossible that the line be infinite [sit linea infinita]. It does not, however, belong to geometers to judge concerning these words, because this is appropriate only for a teacher of natural science; geometers, however, now and then posit that lines are infinite [ponunt lineas esse infinita]. Furthermore, a circumflex line is infinite [infinita]. Euclid, howe-

ver, did not want to mean anything except that finite lines are ended with points in the same way that surfaces are ended by lines and ... just as all that which is of one genus is ended by that which is less than it by one dimension” [cf. LoBello 2003b, p. 16].

The comparison of the above fragment from the Leiden MS with the relevant text of the Qom Ms brings into light a difference: the Qom fragment is “full of motion”, which means that Simplicius (applying Aristotle’s philosophy in mathematics) and the Arabs thought of geometry in somewhat changed intuitive model. Moreover, in the Arabic text there are words “bounded” and “unbounded” instead of “infinite”. An English translation of the fragment is: “Just as a line, when it moves from its position and produces a surface, so the extremities of the line, when they are set in motion, produce thereby the lines enclosing the surface. He [i.e. Simplicius – Z.K.] means that when the line moves from its position and produces a surface, two extremities are produced for the surface; the two extremities of the line produce the two of them by the motion of the two of them in association with its movement. and as for the two remaining extremities, one of the two of them is the first position of the line, and the second is the position at which it ends. And that is because the statement of Euclid here concerns the bounded surface and not an unbounded or a spherical surface” [cf. LoBello 2009, pp. 3–4].

From the Simplicius’ fragments preserved in the Arabic and Latin texts of al-Nayrizi’s commentary, it is clear that Simplicius locates the realm of mathematical objects in the realm of imagination. Let us remind the reader that for Aristotle every geometrical object has to be represented by a real property of a real, physical object (substance, body) Simplicius and his followers: the Arabs, Gerard, Albertus Magnus, can see that postulates may be not realizable in the real world. Therefore, Simplicius and the Arabs introduce the concept of an imagined mathematical object. The conflict with reality is especially sharp with respect to infinite objects even if they are thought as only potentially infinite. The realm of mathematics *exceeds* the reality. For Simplicius and medieval authors, this “exceed” is apparent only in some secondary points, mainly concerning the possibility of unbounded extension of some objects, mainly some lines and surfaces. However, this transgression of reality is seen as the main obstacle in understanding of geometry by students and people uneducated in geometry. The above partial and seemingly harmless (because only imagined), *separation* of geometrical objects from the reality allowed al-Nayrizi to operate with the *two* different concept of lines: finite and *actually infinite*. Though there is no infinite line in reality, it can nevertheless exist in pure imagination. The above ideas are crucial, therefore, they have to be supported with some sources. Firstly, let us indicate some fragments concerning the realm of imagined objects and the role of imagination.

The Ms Qom sets the phrase: “[H]e has certainly distinguished it [i.e. a circle – Z.K.] from the plane surfaces that do not form a figure. like the plane surfaces that are imagined to be unbounded, or those bounded on some sides and unbounded on other sides, and he also distinguished it from lines and solids” [cf. LoBello 2009, p. 10].

The same fragment in the Latin translation employs the concept of infinity in somewhat different sense from the original Greek concept of indetermination: “[I]t is separated from figures which are shapeless, like the surfaces which are imagined to be infinite [que imaginatur infinite], and others which are on one side finite and on the other infinite [ab alia infinite] and he has also separated it from lines and bodies [...]” [cf. LoBello 2009, p. 30, Simplicius’ comment concerns Def. I.14 in the Tummies’ numbering, and Heiberg-Curtze 17.5–10; the fragment is not quoted by Albert].

The same gradual change of the meaning of the concept of infinity is seen, for instance, in the comments to Def. I.16. The Ms Qom uses the phrase: “And if the perpendicular on the center of the circle should be extended in both directions indefinitely [...]” [cf. LoBello 2003a, p. 11]. The same fragment in the Latin is: “[...] but if the perpendicular that is above the center be drawn from each side to infinity [ab utraque parte in infinitum protrahatur]” [cf. LoBello 2009, p. 31, Heiberg-Curtze 18.4–5]. Also, the definitions of parallel lines contain the same phrases: “Parallel straight lines are those that are in one plane, and if they are extended on each of their two sides without bound, do not meet, not on any of the two sides” [Ms Qom cf. LoBello 2009, p. 16]. “Equidistant straight lines are those which, although they are on the same surface, if they are extended on either side, even in infinitum [si utique etiam in infinitum protrahantur], will not run together on either of the two sides” [Gerard, cf. LoBello 2003b, p. 39, Heiberg-Curtze 25.5–8].

The same, small difference is seen in the comments to the above definitions of parallel lines. The Arabic manuscripts (Ms Qom) have: “[...] if these two lines are now both extended indefinitely in each direction” [Simplicius, Ms Qom; LoBello 2009, p. 16], “if they are extended with an endless extension” [Aganis, Leiden Ms; LoBello 2003a, p. 88]. The corresponding Latin fragments in Gerard are: “even if they be extended in infinitum [etsi in infinitum protrahantur]” [cf. LoBello 2003b, pp. 39–40; Heiberg-Curtze 25.24], and “if they should be extended in infinitum on both sides [si utique in infinitum protrahantur]” [cf. LoBello 2003b, p. 40; Heiberg-Curtze 26.14–15]¹⁰. Albert has a little different formulation of the definition of parallel lines: “Equidistant lines are those which, located on the same surface and extended on each side, do not come together even if

¹⁰ The Latin manuscripts contain one more usage of the Aganis-type Latin phrase, which is not transmitted by the Arabs; cf. Heiberg-Curtze 25.30: “in infinitum protracte fuerit”.

extended in infinitum”. He has also in the comments: “they should be extended in either direction in infinitum” [cf. LoBello 2003c, p. 22]. Aganis’ definition of parallel lines is the same as in the Leiden Ms.

From the above, it is clear that the parallel lines are finite lines which can be extended indefinitely “on both sides”. However, in all the above commentaries, certain intriguing novelty emerges in comparison to the ancient Euclid’s geometry. The Arabic as well as the Latin sources preserve additional al-Nayrizi’s explanation: “As for his [i.e. of Aganis – Z.K.] statement »if the two of them are extended with an endless extension, infinitely«, he only said that for imagination’s sake, in order not to force a restraint on the two of them, for this reason: not that their extensions pass beyond the sphere of the fixed stars, but in order that it should not happen that if we posited segments for their extension, then they would not meet on what we allocated for the two lines, but that it would be possible for them to meet if they passed beyond the that boundary; the two of them would certainly not meet. This is what was commonly said about this obstacle, only it is an abbreviation and a summary of what others said on the subject at greater length” [cf. Leiden Ms, LoBello 2003a, p. 89].

The corresponding Latin fragments are: “As for the fact that he said »they may be extended in infinitum«, he did not say it except insofar as concerned the imagination – for both would be wanting, since their extension would be in a space which would be greater than the space which is between us and the sphere of the fixed stars – but in order that there might be, when we shall have posited their extension at any boundary where they are not joined, that which is beyond, where they are not joined, and that we might indicate that they are not joined. This, too, was the custom right up to now in this matter, that they would posit this to avoid a multitude of words and to lay hold of brevity” [cf. LoBello 2003b, pp. 41–42; Heiberg-Curtze 27.14–27]. “As for the fact, moreover, that he says »extended in infinitum«, he means only according to the imagination, and not according to the being of infinite space” [cf. Albert, LoBello 2003c, p. 23].

Thus, Albertus Magnus operates in geometry with the concept of *imaginary* infinite space and such an object does not exist in the real world. The concept of an infinite space is absent in other Arabic and Latin sources. Albert, in an unintended way, *changes* the original meaning of al-Nayrizi’s remark.

More information about the imaginary realm of infinite geometric objects is given in the sources in the introductory remarks to Euclid’s postulates. From these comments preserved in the Latin sources of Gerard’s translation, one can see that the original author of the above al-Nayrizi’s remarks is probably Simplicius. However, in the *Manuscript Leidensis* the comments are evidently attributed to al-Nayrizi. The *Codex Leidensis*, after the explanation that Euclid’s postulates are difficult for a student because they, “in a word, are what are not established” and they are “sometimes impossible” [i.e. in the real world],

explains that they are similar to one Archimedes's postulate in which Archimedes conceded that it is possible for him to stand outside the earth. "Now, this was the result of his boasting of having discovered the power of geometry. So he requested that it be postulated thus, and it was so granted for the purpose of instruction, even though it was impossible" [cf. LoBello 2003a, p. 90]. In the same way "it is certainly not possible for me to draw a straight line on the surface of the sea" or "it is not possible for me to extend a straight line without limitation, infinitely, since the infinite does not exist" [Gerard has: "for infinity cannot be found"; "infinitum enim non reperitur"; cf. Heiberg-Curtze 27.26; cf. *ibidem*, p. 91]. Thus, the postulates are necessary for "the transmission of knowledge"¹¹. A similar (but longer) fragment is attributed to Simplicius in the Gerard's Latin translation.

Al-Nayrizi postulates the existence of *geometrical matter*: "As for this postulate [i.e. the first – Z.K.], it is necessary to ask that it be postulated because the existence of geometrical matter is in the imagination. For, indeed, if their existences were in material bodies, it would be rash to postulate that a straight line be drawn from Aries to Libra" [cf. *Codex Leidensis*, LoBello 2003a, p. 92]. "And this by necessity had to be posited, because the existence of geometrical matter consists in the imagination [quod essentia materie geometrie consistit in imaginatione]. For if it were in bodies having matter, it would be superfluous that it be asked to be postulated that a straight line be drawn from Aries to Libra" [cf. Gerard, LoBello 2003b, p. 45; Heiberg-Curtze 31.1–5].

Albert is more explicit in saying that there is no "geometrical matter" because it is an unacceptable thing in his nominalistic Aristotelian philosophy: "Deceived, therefore, are they who said that they [i.e. the postulates – Z.K.] are postulated for no other reason than that geometrical matter be generated through them, namely, because all geometry revolves around imaginable quantity and not the sensible continuum" [cf. LoBello 2003c, pp. 23–24].

It seems that al-Nayrizi uses two concepts of line: finite and infinite. Gerard translates the Arabic text as follows: "As, however, for the fact that in the definition [of the second Euclid's postulate – Z.K.] it is added that it is a finite line, it has been well said, since if it were an infinite line, it could not be extended. Moreover, it is possible that a finite line be extended in infinitum, if it should be necessary, which is done for this reason, lest the shortness of the lines impede us in certain figures [i.e. in certain theorems – Z.K., because every theorem is called a "figure" since every one had an attached to it diagram]" [cf. LoBello 2003b, p. 46, Heiberg-Curtze 31.15–20].

Codex Leidensis is probably incorrectly translated by Lo Bello who reads: "We know why it says in the definition that the line is finite for, indeed, if it were

¹¹ This last view is attributed to Boethius by Albert; cf. also Boethius *Quomodo substantiae*, P.L. 64, 1311b.

infinite, how would it be possible for it to be extended? And as for the finite line, it is now posited that its extension be infinite if necessary; this is in order that the shortness of the line not confine us in any of the figures” [cf. LoBello 2003a, p. 93; Lo Bello thinks that the “figure” means “geometrical figure”, not a “theorem”].

Such comments were next to impossible in the Euclid’s times because there is a big difference between an unbounded, indeterminate line and the (actually) infinite line. (Albert omits these comments in his *Commentary*.)

Al-Nayrizi indicates also some new axioms which were introduced to geometry after Euclid. Pappus is counted among the developers of new axioms. One of his axiom is: “We shall need this result in the first figure [i.e. in the first theorem – Z.K.]: With regard to the straight line and the plane surface, it is possible, because of their levelness, for them to be extended with an infinite extension, forever” [cf. LoBello 2003a, p. 104; cf. also Proclus’ *Commentary on the first book of Euclid’s Elements*, 198.6–10. The above comment is explicitly attributed to Pappus by al-Nayrizi]. “[A]nd it is possible for a plane surface and a straight line, for this reason, because they are plane, to be extended in infinitum [in infinitum protrahi] [cf. Gerard, LoBello 2003b, p. 54, Heiberg-Curtze 38.10–15; Pappus is not explicitly indicated].

Albert adds that “infinitely many others [i.e. common notions – Z.K.] can be added” [cf. LoBello 2003c, p. 30] and he lists some of them. He writes: “For magnitude decreases in infinitum. Among numbers, however, if the prior should be a submultitude of the second, whatever third will be equally a submultitude of some fourth. Multitude increases in infinitum”

The next part of al-Nayrizi’s *Commentary* concerns the theorems of the *Elements*. Gerard – contrary to the Arabic sources containig al-Nayrizi’s *Commentary* – usually does not translate Euclid’s formulation of the theorems and proofs. He restricts himself almost only to the comments and some alternate variants of the proofs. As it was in the case of Euclid himself, the Arabic as well as the Latin comments form the evidence of ancient finitism. The lines are extended “according to straightness” (e.g. theorems IL.1¹², IA.2¹³, IA.16, IA.29, IA.37 or IA.38¹⁴), usually to the definite points; cf. for instance the theorems IA.5, IA.7, IL.7, IL.11, IA.16, IA.21, IA.25, IL.25, IA.31, IA.32, IA.44 (3 times), IA.46, IL.46 (4 times), IL.47. Also, in Albert’s *Commentary* which

¹² I designate by “I.A.xx” a theorem in an Arabic source, and by “IL.xx” a theorem in a Latin source.

¹³ Gerard has in his translation the following phrase in a variant of the proof of the theorem I.2: “Then I shall extend the two lines BD and DA according to straightness, nor shall I posit a boundary to their extension, until they are so long that when a circle shall be described, there may remain something left over from each one of them”; cf. LoBello 2003b, p. 62–63.

¹⁴ In IA.38 indefinite extension is used without explicit naming of the fact.

contains Euclid's text, the extensions are used, e.g. I.16, I.17, and extensions usually end at the determined points; cf. the following theorems: I.1 (also in Heron's variants), I.2, I.3, I.5, I.6, I.7, I.16 (very interesting cases of the extension), I.19 (in Heron's variant), I.20, I.22, I.23, I.27, I.38, I.39 (3 times), I.40, I.42, I.44, I.46 (many times, also in Heron's variants).

However, for us, the most important are some candidates for *infinite* extensions of lines. It is not only a philological problem of the use of some names designating "infinity", "infinite extension", etc. More essential is the general attitude to the problem of infinity. It appears that in *every* place in the sources, even if the word "infinite" or "infinity" is used, every extension is expressed within the ancient sense of the words "finite" but "indefinite", i.e. of the length not strictly determined. Moreover, finitism is more dominant in younger sources. The most explicit finitism is found in the Albertus Magnus' *Commentary*.

Coming back to the details, one can list the places in the sources at which indefinite extensions are used. There are two groups of such extensions. The first group is created by the mentioned above already first group of the extensions which do not terminate in a definite point or points, e.g. IA.2, IL.1, etc. In every such place, one can see from the proof or a general context that the extensions in reality are finite and that the lines are extended up to the point where the given problem or proof can be done. In some places, however, there are phrases which would suggest that the extension is infinite. For instance, in the theorem IA.11, in one Heron's variant of the proof, one finds the words: "let us draw the perpendicular GD to it [...] and let its extension be without limit, and let us cut off GD equal to line AG [...]". The context of the proof, however, indicates that the "infinity" of the line GD is irrelevant for the reasoning. GD has to be simply "so long" that the cutting off AG is realizable, i.e. that, simply, GD must be longer than AG. The point A is given as well as the line AB. However, previously we had to choose a random point G on AB. Thus AG is not exactly determined. Nevertheless, we know that the line AG is shorter than the line GD. Therefore, it is possible to reformulate the proof without the use of any indefinite extension of GD and supposing instead that the line GD is no less than the line AB. However the last move is done in "modern (i.e. metrical) wording" of geometry. The same Latin variant of the proof of I.11 contains the word "infinite": "And so let the drawing of GD be in infinitum [Sit itaque protractio GD in infinitum]. I shall, moreover, cut from GD what is equal to line AG" [cf. LoBello 2003b, p. 71; Heiberg-Curtze 55.4–5]. Albert has a little changed variant of the proof in which there are *two* lines of indefinite length (indefinite quantitatis) [cf. LoBello 2003c, p. 60 and Tummers 1984]. The same variant of Heron's proof is preserved in the **V-B** manuscript but it is absent from Robert of Chester's edition who, however, uses the expression "linea aliqua quantitatis indefinite"[cf. Busard 1996a, p. 57]. The proof is also absent

from one other Albert's source, i.e. from Adelard II version in which there is no proofs but there are only some indications to the proofs; cf. however the formulation of the theorem I.12 in Adelard II where the same expression (indefinite quantitatis) is in use [see Busard 1996, vol. I, p. 113].

The other examples of the extensions of the first group are the theorems I.16 and I.17 or I.32 which are absent from the Gerard's text but they are present in that of Albert. Theorem I.16 has in its formulation the expression "one of its sides ... is extended" [cf. LoBello 2003a, p. 135], however, in the both theorems it is seen from the context of the proofs that the extensions are simply finite and determined (I.32) or finite but indefinite (I.16, I.17). The same one can say about Albert's Latin formulation of the theorems ("one of the sides of a triangle is extended straightly" [quodlibet laterum trianguli directe protrahatur])¹⁵.

The second group of extensions of lines is created by the extensions of parallel lines. This case is the most important. First of all, *every* parallel line is finite and has determined endpoints. The relevant theorems which speak about the parallels are: IA.27*, IA.28 (which contains also an Aganis' proof), IA.29*, IA.30, IA.31*, IA.33, IA.34, IA.35, IA.36, IA.37*, IA.38*, IA.39, IA.40, IA.41, IA.46 (the extensions of parallel lines are present in the theorems marked with "*"); the remaining theorems contain only the evidence that parallel lines are finite). The corresponding Latin (Gerard of Cremona's) theorems are the following: the comments to Postulate V* (also with Agapius' reasoning), IL.31*, IL.38, IL.46. Albert has in this group the theorems: I.27*, I.28, I.29*, I.30, I.31, I.32, I.33, I.34, I.35 (with Postulate 4* of Agapius), I.37, I.38*, I.39*, I.40, I.41, I.42*, I.43, I.44*, I.45, (Heron's variants of) I.46.

IA.27* contains the expression "if they are extended in both dimensions together, will not meet even if they are extended without limit", IA.29*: "and let us extend line ZD with an extension without limit" (this extension is indeterminate, however, ZD is extended only in due to reach the point Q of the intersection of ZQ and one other line; IA.28* contains also the repetitions of the definition of parallel lines in which there are used expressions of the type "an endless extension"), IA.31*, IA.37*, IA.38*: contain finite extensions running to a determined point or finite but not strictly determined.

Gerard of Cremona's text contains the following expressions (in the comments to Postulate V*): there is a repetition of the definition of parallels with the expression "if they are extended on both sides in infinitum", there is also used one finite extension to the determined point, and two infinite extensions, one in the proof of Agapius ("I shall draw line ZN in infinitum"), and the second one in the repetition of Agapius' definition of parallels. However, in the case of that proof, the infinite

¹⁵ Cf. also the same wording in **V-B**, Robert of Chester, and Adelard II. Cf. also the Latin translation of the *Elements* by Gerard of Cremona: cuius libet trianguli latus in rectitudine protrahatur; see Busard 1984, p. 13.

extension is necessary for the performance of the construction required by the proof, and, in reality, the extension is indefinite and potentially infinite. In the same way, Gerard speaks about the possibilities of divisions of a line in the proof of the next theorem, (IL.30), i.e. that we can perform a given constructions as many times as we wish to.

In Albertus Magnus' comments, finite but indefinite extensions are used many time. The only candidate for infinite extension of the parallels is in the theorem I.35 in Agapius' proof of Postulate 4*: "I shall draw from point F a line equidistant from line AB, which let be FG, and I shall draw it in infinitum, for however much distance as the thirty-first [theorem] of Euclid teaches". Thus, in Albertus Magnus, it is evident from the context that the above extension is indeterminate but finite.

From the analysis of the above cases, one can conclude that the actual infinity of the extensions is irrelevant for the proofs and that the authors had in mind only potentially infinite objects.

A separated case of the use of the concept of infinite line concerns the formulations and proof of the theorem I.12 in which the expression "infinite line" is used explicitly.

Al-Nayrizi writes (theorem IA.12): "We want to demonstrate how we may draw, from an assigned point to a known, limitless straight line, a line that is perpendicular upon it. [...] [T]he limitless straight line is line AB. [...] So, we have drawn from the known point G to the line AB, whose magnitude is not known, the line GH, perpendicular to it [...]" [cf. LoBello 2003a, p. 129–130].

There is no Gerard version of this theorem. However, in his translation of Euclid's *Elements*, the theorem is formulated as follows: "Ad lineam rectam infinitam datam a puncto extra eam dato lineam rectam que sit super eam perpendicularis ducere. [...] Sit linea recta data que est non finita *ab*, punctum quoque datum non existens super eam punctum *g*. [...] Iam igitur ad lineam *ab* rectam datam que est infinita [...]" [cf. Busard 1984, p. 11].

Albert is much more explicit: "From an assigned point off of a given line of indefinite quantity, to draw a perpendicular.

This proposition posits that the given line is of indefinite quantity for this reason that it may be everywhere be under the given point, since if it were otherwise, it would not be possible to proceed to a proof. Either, therefore, the point will be given opposite the middle of the line, where »middle« may be call whatever is between its extremes, or opposite or above an extreme point" [cf. LoBello 2003c, pp. 62–63]. The formulations of the enunciations of this theorem by Robert of Chester, the **V-B**, and the Adelard II are the same [cf. Busard 1996a, p. 58; Busard 1996, p. 119. Cf. also the same comment concerning the indefinite line in this theorem in Proclus' *Commentary on the first book of Euclid's Elements*, 284–286].

In summary: the infinite line in the theorem I.12 is finite but of indefinite length. Moreover, the given indefinite line does have extreme points.

A general conclusion of the present section is that the ancients were finitists but in some commentaries by Simplicius and the Arabs, the concept of an imaginary infinite line and surface emerges. The Arabs accepted also a special kind of *imaginary mathematical matter* from which geometrical objects are created by the mind. Albert uses also a concept of space.

Now we can consider the imaginary infinite space in physics, astronomy and theology in the Middle Ages as described by E. Grant [1982], Chapter 6, *Late medieval conceptions of extracosmic ("imaginary") void space*. A progenitor of an imaginary infinite space is, obviously, Aristotle; cf. for instance his *Physics* 203b. The Arabic commentators, as Averroes, also discussed this concept, however, they, as Aristotle did, rejected the real existence of extracosmic imaginary space, accepting at the same moment the existence of some infinite geometrical object made of a special kind of matter. Also, Robert Grosseteste, Thomas Aquinas, Pseudo-Siger of Brabant rejected the extracosmic space. For them, what is in imagination does not exist. However, Nicole Oresme accepted the real existence of extracosmic void [cf. Grant 1982, p. 119]. His views are also a turning point for the use of actually infinite objects in mathematics and *mathematical problems* in astronomy: he was the first. However, I postpone the analysis of the Oresme's (J. Wallis', J. Kepler's et al.) works to another work.

As it was already explained, the development and the application of the infinite concepts in mathematics is not directly connected with the emergence of the infinite notions in philosophy, theology or general astronomy¹⁶.

Bibliography

- Bacon R., *Opera hactenus inedita Rogeri Baconi*, Fasc. 8: *Questiones supra libros quatuor Physicorum Aristotelis*, F.M. Delorme O.F.M., R. Steele (eds.), Clarendon Press, Oxford 1928.
- Bergh S. van den, *Averroes' Tahafut al-Tahafut (The Incoherence of the Incoherence)*, Oxford University Press, Oxford 1954, Luzac, London 1954.
- Besthorn R.O. et al., *Codex Leidensis 399,1. Euclidis Elementa ex interpretatione al-Hadschschadschii cum commentariis al-Narizii*, Copenhagen 1893–1932.

¹⁶ “Long before Oresme, al-Razi (ca. 854–925 or 935), or Rhases argued that reason told even simple folk that empty space exists outside the world. [...] For Abu'l-Barakat al-Bagh-dadi (ca. 1080 – d. after 1164/1165), reason or the estimative faculty was responsible for the innate knowledge in the human mind that empty three-dimensional space filled with bodies is prior to the notion of a plenum [...]”. Cf. Grant 1982, p. 322, footnote 18. “Crescas may have been the first scholar in Western Europe since Greek antiquity to have adopted unequivocally the existence of an infinite three-dimensional void space”. Cf. Grant 1982, p. 322, footnote 20.

- Boethius, *Anicii Manlii Torquati Severini Boetii De institutione arithmetica libri duo, De institutione Musica libri quinque, accedit Geometria que fertur Boetii et Libris manu scriptis*, G. Friedlein (ed.), B.G. Teubner, Lipsiae 1867.
- Boethius, *Boethius und die griechische Harmonik. Des Anicius Manlius Severinus Boethius „Fünf Bücher Über die Musik“*, O. Paul (ed.), Verlag von F.E.C. Leuckart (Constantin Sander), Leipzig 1872.
- Brentjes S., *The relevance of non-primary sources for the recovery of the primary transmission of Euclid's Elements into Arabic, Tradition, Transmission, Transformation*, Norman, Oklahoma 1992/1993. Collection de Travaux de l'Académie Internationale d'Histoire des Sciences 37, Brill, Leiden 1996, pp. 201–225.
- Brentjes S., *Textzeugen und Hypothesen zum arabischen Euklid in der Überlieferung von al-Haggag b. Yusuf b. Matar (zwischen 786 und 833)*, “Archive for History of Exact Sciences” 1994, 47, pp. 53–92.
- Brentjes S., *Two commentaries on Euclid's Elements? On the relation between the Arabic text attributed to al-Nayrizi and the Latin text ascribed to Anaritius*, “Centaurus” 2001, 43(1), pp. 17–55.
- Busard H.L.L., *The translation of the Elements of Euclid from the Arabic into Latin by Hermann of Carynthia(?) : books I–VI*, E. J. Brill, Leiden 1968.
- Busard H.L.L., *The Translation of the Elements of Euclid from the Arabic into Latin by Hermann of Carynthia(?) : Books VII–XII*, Mathematisch Centrum, Amsterdam 1977.
- Busard H.L.L., *The First Latin Translation of Euclid's Elements Commonly Ascribed to Adelard of Bath*, Pontifical Institute of Mediaeval Studies, Toronto 1983.
- Busard H.L.L., *The Latin translation of the Arabic version of Euclid's "Elements" commonly ascribed to Gerard of Cremona*, E.J. Brill, Leiden 1984.
- Busard H.L.L., *Some early adaptations of Euclid's Elements and the use of its Latin translations*, (in:) *Mathemata. Festschrift für Helmut Gericke*, Reihe “Boethius”, M. Folkerts, U. Lindgren (eds.) Bd. 12, Franz Steiner Verlag Wiesbaden GmbH, Stuttgart 1985.
- Busard H.L.L., *The Medieval Latin Translation of Euclid's Elements Made Directly from the Greek*, Franz Steiner Verlag Wiesbaden GmbH, Stuttgart 1987.
- Busard H.L.L., Folkerts M., *Robert of Chester's (?) redaction of Euclid's Elements, the so-called Adelard II version*, vols. I, II, Institute für Geschichte der Naturwissenschaften, München 1996 (also: Birkhäuser Verlag, Basel – Boston – Berlin 1992).
- Busard H.L.L. (ed.), *A Thirteenth-Century Adaptation of Robert of Chester's Version of Euclid's Elements*, vols. I, II, Institute für Geschichte der Naturwissenschaften, München 1996.
- Busard H.L.L., *Johannes de Tinemues redaction of Euclid's Elements, the so-called Adelard III version*, vols. I, II, F. Steiner, Stuttgart 2001.
- Busard H.L.L., *Campanus of Novara and Euclid's "Elements"*, vols. I, II, F. Steiner, Stuttgart 2005.
- Clagett M., *The Medieval Latin Translations from the Arabic of the Elements of Euclid with Special Emphasis on the Versions of Adelard of Bath*, “Isis” 1953, 44, pp. 27–28, 38–42.
- Cornford F.N., *The invention of space*, (in:) *Essays in Honour of Gilbert Murray*, Allen & Unwin, London 1936.
- Czwalina A., *Kleomedes Die Kreisbewegung der Gestirne*, (in:) *Ostwald's Klassiker der exacten Wissenschaften*, Engelmann, Leipzig 1927.
- Folkerts M., *Boethius' Geometrie II: Ein mathematisches Lehrbuch des Mittelalters*, Wiesbaden 1970.
- Folkerts M., *Ein neuer Text des Euklides Latinus. Faksimiledruck der Handschrift Lüneburg D 4o 48, f 13–17v*, Hildesheim: H.A. Gerstenberg 1970a.
- Folkerts M., *Euclid in Medieval Europe*, The Benjamin Catalogue for History of Science, University of Winnipeg (1989). Retrived 2012 [online] <www.math.ubc.ca/~cass/euclid/folkerts/folkerts.html>.

- Geymonat M., *Euclidis Latine facti fragmenta Veronensia*, ed. M. Geymonat, Istituto Editoriale Cisalpino, Milano, Varese 1964.
- Geymonat, M., *Nuovi frammenti della geometria "boeziana" in un codice del IX secolo?*, "Scriptorium" 1967, 22, pp. 3–16.
- Grant E., *Much Ado about Nothing. Theories of Space and Vacuum from the Middle Ages to the Scientific Revolution*, Cambridge University Press, Cambridge – London – New York – New Rochelle – Melbourne – Sydney 1982.
- Heath T.L., *The thirteen books of Euclid's "Elements" translated from the text of Heiberg with introduction and commentary*, vols. 1–3, University Press, Cambridge 1908. Retrieved 2008, [online] <www.wilbourhall.org/pdfs/>.
- Heiberg J.L., *Euclidis Elementa*, Teubner, Leipzig (1883–1888). Retrieved 2007, [online] <www.perseus.tufts.edu/cgi-bin/ptext?lookup=Euc.+toc> or <www.wilbourhall.org/>.
- Heiberg J.L., *Euclidis opera omnia*, (in:) J.L. Heiberg, H. Menge (eds.), *Bibliotheca Scriptorum Graecorum et Romanorum Teubneriana*, vols. I–VIII, Aedibus B.G. Teubneri, Lipsiae 1883–1889. Retrieved 2009, [online] <www.wilbourhall.org/>.
- Król Z., *Platon i podstawy matematyki współczesnej. Pojęcie liczby u Platona, (Plato and the Foundations of Modern Mathematics. The concept of Number by Plato)*, Wydawnictwo Rolwski, Nowa Wieś 2005.
- Król Z., *Scientific Heritage: The Reception and Transmission of Euclidian Geometry in Western Civilization*, "Dialogue and Universalism" 2012, 4, pp. 41–66; cf. also the text [online] <<http://calculemus.org/pub-libr/index.html>>.
- Kunitzsch P., *Findings in Some Texts of Euclid's "Elements" Mediaeval Transmission, Arabo-Latin*, (in:) *Mathemata. Festschrift für Helmuth Gericke*, ed. M. Folkerts, U. Lindgren, Franz Steiner, Stuttgart 1985, pp. 115–128.
- Lo Bello A., *The Commentary of al-Nayrizi on Book I of Euclid's The Elements of Geometry with an Introduction on the Transmission of Euclid's Elements in the Middle Ages*, "Ancient Mediterranean and Medieval Texts and Contexts; Medieval Philosophy, Mathematics, and Science" 1, ed. A. Lo Bello, 2003a.
- Lo Bello A., *Gerard of Cremona's Translation of the Commentary of al-Nayrizi on Book I of Euclid's Elements of Geometry With an Introductory Account of the Twenty-Two Early Extant Arabic Manuscripts of the Elements*, "Ancient Mediterranean and Medieval Texts and Contexts; Medieval Philosophy, Mathematics and Science" 2, ed. A. Lo Bello, 2003b.
- Lo Bello A., *The Commentary of Albertus Magnus on Book I of Euclid's Elements of Geometry*, "Ancient Mediterranean and Medieval Texts and Contexts; Medieval Philosophy, Mathematics and Science" 3, ed. A. Lo Bello 2003c.
- Lo Bello A., *The Commentary of al-Nayrizi on Book I of Euclid's Elements of Geometry With a Translation of That Portion of Book I Missing from MS Leiden Or 399.1 but Present in the Newly Discovered Qom Manuscript Edited by Rüdiger Arnzen*, "Ancient Mediterranean and Medieval Texts and Contexts; Medieval Philosophy, Mathematics, and Science" 8, ed. R.M. Brchman, J.A. Finamore, Brill, Leiden – Boston 2009.
- Lorch R., *Some Remarks on the Arabic-Latin Euclid*, (in:) *Adelard of Bath. An English Scientist and Arabist of the Early Twelfth Century*, ed. Charles Burnett, The Warburg Institute 1987, pp. 45–54.
- Murdoch J.E., *Euclides Graeco-Latinus: A Hitherto Unknown Medieval Latin Translation of the Elements Made Directly from the Greek*, "Harvard Studies in Classical Philology" 1967, 71, pp. 249–302.
- Murdoch J.E., *The Medieval Euclid: Salient Aspects of the Translations of the Elements by Adelard of Bath and Campanus of Novara*, (in:) *XIIIe Congres International d'Histoire des Sciences*, Revue de Synthese, 3e série, 49–52, Paris 1968.
- Mynors R.A.B., *Cassiodori Senatoris Institutiones*, ed. R.A.B. Mynors, 2nd ed., Oxford 1961.

Proclus, *A Commentary On the First Book of Euclid's Elements*, G.R. Morrow (ed.), Princeton University Press, Princeton, New Jersey 1992.

Tummers P.M.J.E., *Albertus (Magnus)' commentaar of Euclides' "Elementen" der geometrie, deel II*, Nijmegen 1984.

Tummers P.M.J.E., *Anaritus' commentary on Euclid. The Latin translation, I-IV*, Ingenium Publishers, Nijmegen 1994.

Ziegler H., *Cleomedes, De motu circulari corporum caelestium*, Ziegler H. (ed.), Greek text and Latin translation, Teubner, Leipzig 1891.