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JUMP-DIFFUSION AND GARCH OPTION PRICING MODEL – A REVIEW OF THEORY AND EMPIRICAL STUDY FROM DTB

Abstract

Using transaction data on the DAX index options, we study the jump-diffusion option pricing model and an option pricing model under fairly nonlinear generalized autoregressive heteroskedasticity N-GARCH (1,1). The paper also contains an overview of some of jump-diffusion process estimation methodologies and a survey of empirical research on the time series of DAX index and series of DAX options. Our empirical evidence indicates that adding the random-jump feature to the random walk model of asset return outperforms the N-GARCH pricing option model in valuing short-term options, whereas the N-GARCH model does better than the jump-diffusion model in pricing longer-term options.

Introduction

Since Black and Scholes published their seminal article on option pricing in 1973, there has been an explosion of theoretical and empirical work on option pricing. Most papers maintained Black and Scholes' assumption of geometric Brownian motion, i.e that asset prices are stationary and log-normally distributed. However the accuracy of the hypothesis has already been questionable for ages. Early empirical studies of market stock prices (reported in Mandelbrot 1963, Fama 1965, Praetz 1972, among others) concluded that the log-normal law is an inadequate descriptor of stock returns. It is widely acknowledged that many financial markets exhibit a greater degree of kurtosis and the presence of non-zero skewness than is consistent with the geometric Brownian motion model of Black and Scholes. It is also well known that implied volatilities obtained from option prices under the Black-Scholes model are typically higher for out-of-the-money options than for otherwise identical at-the-money option (the so-called "volatility smile"). It is also typically the case that the differences in implied volatilities become less pronounced as options with a greater time-to-maturity are considered. The existence of the "volatility smile", as Bates (1996) and others observed, is strongly suggestive of the presence of leptokurtosis in the conditional return distributions. Evidence of conditional leptokurtosis also comes from the ARCH/GARCH literature in the form of fat-tailed residuals[1].

Among the many alternative models that have been proposed in the finance literature to account for these systematic departures from the assumptions of the Geometric Brownian Motion model, two have become especially popular in recent years: models of jump-diffusions, and models of stochastic volatility. Models of jump-diffusions i.e. models in which stock prices are generated by a geometric Brownian motion with Poisson jumps superimposed on them, were initially proposed by Merton(1976); more recently, they have been studied by Ball and Torous (1985), Ahn (1992), Amin (1993), among others. The stochastic volatility models also involve a generalization of the Geometric Brownian Motion process by allowing the volatility of the return process to itself evolve stochastically over time. Recent papers adopting such an approach include Wiggins (1987), Melino and Turnbull (1990), Stein and Stein (1991), Amin and Ng (1993), Heston (1993) and others. In addition, Joion (198), Bates (1996) and Cao, Bakshi, Zhiwu Chen (1997) have empirically estimated models of returns as mixtures of jump-difusions combined with stochastic volatility.

At a purely intuitive level, it is not hard to check how jump-diffusion and stochastic volatility models could each lead to return distributions that exhibit skewness and leptokurtosis. Skewness in jump-diffusions should arise from the distribution of jump-sizes. Moreover, the presence of jumps in the return process creates outliers which add fatness to the tails of the distribution; thus, returns under a jump-diffusion should always be leptokurtic. In stochastic volatility models, skewness should arise if there is non-zero correlation between the stochastic processes driving changes in the returns and the volatility respectively; for instance, positive correlation between these process should result in positive skewness, since high returns will be accompanied by high volatility, and low returns by low volatility. Leptokurtosis should arise from the volatility of volatility.

Empirically, however, it is not known whether and by how much each generalization improves option pricing. The purpose of the present article is to fill in this gap and conduct a comprehensive empirical study of the relative merits of two option pricing models.

Option pricing models: Jump-diffusion and N-GARCH (1,1) model

Typically, the option pricing models in a continuous-time framework have employed special cases of the following general system of stochastic differential equations of the form:

$$dS_t = [\mu - \lambda\mu_J]dt + \sigma S_t^{\delta} dW_t + J(t)dq(t) \qquad (1)$$

with the volatility

satisfying

$$d\sigma = \mu_{\sigma}(\sigma)dt + v(\sigma)dW_{\sigma} \qquad (2)$$

and the instantaneous nominal discount rate

$$dr = \mu_r(r)dt + v_r(r)dW_r \qquad (3)$$

where

S_t is the option's underlying asset price, with instantaneous expected return μ per unit time;

σ is a volatility state variable;

r is the instantaneous nominal discount rate;

W, W_{σ}, W_r are correlated standard one-dimensional Brownian motions defined on some filtered probability space (Ω, Γ, P);

$J(t)$ is the percentage jump size (conditional on a jump occurring) that is lognormally, identically and independently distributed over time, with the unconditional mean μ_J. The standard deviation of $\ln[1+J(t)]$ is σ_J;

$q(t)$ is a Poisson jump counter with intensity λ, that is, $\Pr[dq(t)=1]=\lambda dt$ and $\Pr[dq(t)=0]=1-\lambda dt$

2δ is the elasticity of variance.

This general specification nests the constant elasticity of variance, stochastic volatility, stochastic interest rate and jump diffusion models. Let us agree to denote the "objective" probability measure which governs the real model (1-3) by the letter P. The fundamental problem in pricing European and American options is the derivation from the *actual* distribution of P the underlying asset price of an *equivalent* "risk-neutral" distribution that summarizes the prices of relevant Arrow-Debreu state-contigent claims. The "risk-neutral" probability distribution is denoted henceforth by the letter Q. Options are then priced at the discounted expected value of its payoff taken under the risk-neutral distribution Q:

$$F(t,S_t) = e^{-\int_t^T r(u)du} E^Q_{t,S_t}\left[\max(|S_T - K|,0)\right] = e^{-\int_t^T r(u)du}\int_0^{\infty}\max(|S_T - K|,0)q_{t,T}(S_T)dS_T \qquad (4)$$

where $q_{t,T}$ is the conditional density function of the random variable S_T representing the underlying asset price at maturity T under the measure Q given the past history up to time t. Thus E^Q denotes expectations taken under the Q-measure. There is a natural economic interpretation of the formula (4). We see that the price of the option, given today's date t and today's underlying asset price, is computed by taking the expectation of the final payment E^Q_t and then discounting this expected value to the present value using the discount factor e. The important point to note is that when we take the expected value we are not to do this using the objective probability measure P. Instead we shall use the Q-measure. This Q-measure is sometimes called the risk adjusted measure, the "risk-neutral" measure or the martingale measure, while the formula (4) is sometimes referred to as the formula of risk neutral valuation. Suppose that all agents are risk neutral. Then all assets will command a rate of return equal to the discount factor $\int_t^T r(u)du$ wz, i.e. in a risk neutral world the stock price will actually have Q-dynamics. Futhermore, in a risk neutral world the present value of a future stochastic payout will equal the expected value of the net payments discounted to present value using the above factor. Observe, however, that we do not assume that the agents in such a model are risk neutral, and these two densities – the historical density P which describes the random variations of the underlying asset price S between t and T, and the "risk-neutral" density Q – are a priori different and, except in very special cases such as geometric Brownian motion and constant elasticity of volatility for which options are redundant assets that can be replicated by a dynamic trading strategy in the underlying asset and a riskless bond, the equivalent risk-neutral distribution can then be derived via no-arbitrage conditions, otherwise the arbitrage arguments do not enable us to calculate one of these given the other.

In other words, the formula (4) only says that the value of the option can be calculated as if we lived in a risk neutral world, and the options cannot be evaluated in such an enviroment by means of arbitrage-free framework pricing without further assumptions. Technically, however, in more complicated cases deriving the appropriate risk-neutral probability measure in those cases could be done but it requires the pricing systematic asset, volatility, interest rate, and jump risk, which in turn requires additional restrictions on distributions and preferences. Intuitively, the market prices for additional risks represent the return-to-risk trade-off demanded by investors for bearing the additional risks of the stock. For such complicated processes, options will usually elude arbitrage-oriented pricing[2]. This process of stock weights eliminating the linear diffusion risk cannot, however, simultaneously remove the non-linear additional risk and vice versa, because the option price is a convex function of the stock price.[3] A non-hedgeable residual risk remains, which one is only able to eliminate via portfolio strategies under very restrictive assumptions (for example existence of a portfolio completely correlated with the residual risk). In such cases, without any additional restictions one can evaluate the option in market equilibrium on the basis of its characteristics as well as on the basis of preferences and endowment of an investor representing the capital market.

In literature of pricing assets some economists such as Hull and White (1987), Johson and Shanno (1987), and Scott (1987) assummed that additional risks are typically nonsystematic and therefore have zero price, or imposed a tractable functional form on the risks' premium with extra (free) parameters to be estimated from observed option prices. The second approache that was used by others such as Wiggins (1987), Melino and Turbull (1990), Bates (1988, 1990), Nail and Lee (1990) is to assume the representative investor has time-seperable power utility, and preferably log utility, so that Cox, Ingersoll and Ross (1985) separability results can be invoked to price the additional risk.

The special case of this general specification was given by Black and Scholes (1973) who remarked that when the price of an underlying asset is described by a geometric Brownian motion process:

$$\frac{dS}{S} = \mu dt + \sigma dW \tag{5}$$

where μ and σ are constant, the price of European call option with strike price K and time of maturity T is given by the formula $\Pi(t) = F(t,S(t))$, where:

$$F(t,s) = sN[d_1(t,s)] - e^{-r(T-t)} KN[d_2(t,s)] \tag{6}$$

Here N is the cumulative distribution function for the $N[0,1]$ distribution and

$$d_1(t,s) = \frac{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma_t^2(T-t)\right)}{\sigma\sqrt{T-t}} \tag{7}$$

$$d_2(t,s) = d_1(t,s) - \sigma\sqrt{T-t} \tag{8}$$

The original Black-Scholes formula has been and continues to be the dominant option pricing model, against which all other models are measured. The dominance of the Black-Scholes model is reflected in the fact that the implied volatility – the value of s that equates to the appropriate option pricing formula to the observed option price – has become the standard method for quoting option prices. However the method used by Black and Scholes relies in an essential way on the hypothesis that the underlying asset follows geometric Brownian motion (4), which does not adequately describe the real dynamics of asset prices.

A proposed jump-diffusion model

The theoretical option pricing formula. Diffusion processes as a model of stock price movements are characterized by infinitesimally small changes, and are for that reason, continuous, although not differentiable, sample paths. Price jumps deviate significantly from this description. On the one hand, they do not vary permanently, but only at certain times. On the other hand, their movement has a non-infinitesimal extent. Jumps are an important feature of the price processes of financial assets, and are especially pronounced for certain types of assets. For that reason, Merton's groundbreaking work, which explicitly admits jumps in the underlying asset prices for pricing standard European options, has generated a profound impact on the finance profession. Merton priced options on jump-diffusion processes under the assumption of diversifiable jump risk and independent lognormally distributed jumps. Subsequent work by, among others, Jones (1984), and Bates (1991) indicates that Merton's model with modified parameters is still relevant even under nondiversifiable jump risk. This section presents a general jump-diffusion model to price options. The model is based on the following assumptions.

Assumption 1: The nondividend-paying asset price $S(t)$, for any t, follows a stochastic differential equation with possibly asymmetric, random jumps:

$$\frac{dS(t)}{S(t)} = [\mu(t) - \lambda\mu_J]dt + \sigma(t)dW + J(t)dq(t) \tag{9}$$

where

$\mu(t)$ – the instantaneous expected return on the asset;

$\sigma(t)$ – the instantaneous volatility conditional on no jumps;

W – the standard Wiener process;

$J(t)$ – the percentage jump size (conditional on a jump occurring) that is lognormally, identically, and independendtly distributed over time, with unconditional mean μ_J. The standard deviation of $\ln[1+J(t)]$ is σ_J. That means:

$$\ln[1+J(t)]\ \text{i.i.d.}N\left(\ln[1+\mu_J] - \tfrac{1}{2}\sigma_J^2, \sigma_J^2\right)$$

λ is the frequency of Poisson events; and $q(t)$ is a Poisson counter with intensity λ:

Pr [one jump occurs in the time interval $(t, t+dt)$] = λdt;

Pr [more than one jump occurs in the time interval$(t, t+dt)$] = $o(dt)$; o(dt) is a function of greater order than dt, such term will be negligible if dt is infinitesimally small;

Pr[no jump occurs in the time interval $(t, t+dt)$] = $1 - \lambda dt$

$\mu(t)$, λ, μ_J, and $\sigma(t)$ are positive.

The equation (9) can be rewritten as:

$$d\ln[S] = \left(\mu(t) - \frac{1}{2}\sigma(t)^2 - \lambda\mu_J\right)dt + \sigma(t)dW + \ln[1 + J(t)]dq(t) \qquad (10)$$

Let $Z(t)$ denote the (continuously compounded) return from holding the asset over the time interval $[t, t+\Delta t]$. Then, by definition:

$$Z(t) = \ln\left[\frac{S(t+\Delta t)}{S(t)}\right]$$

The asset returns $Z(t)$ over the period $[t, t+\Delta t]$ in the above described jump-diffusion process are given by

$$Z(t) = \begin{cases} s(t) & if \quad N(t) = 0 \\ s(t) + \sum_{i=0}^{n(t)} j_i & if \quad N(t) \geq 1 \end{cases} \qquad (11)$$

where

- $s(t) \rightarrow N\left(\left(\mu(t) - \frac{1}{2}\sigma(t)^2 - \lambda\mu_J\right)\Delta t, \sigma(t)^2\Delta t\right)$
- $j_1, j_2, \dots, j_{n(t)} \xrightarrow{i.i.d.} N\left(\ln[1+\mu_J] - \frac{1}{2}\sigma_J^2, \sigma_J^2\right)$. For notational simplicity let

$m = \ln[1+\mu_J] - \frac{1}{2}\sigma_J^2$, and $v = \sigma_J^2$

- $n(t)$ is distributed Poisson with parameter $\lambda\Delta t$. That is, for $n(t) = 0, 1, 2, \dots$ we have

$$\Pr\{n(t) = k\} = e^{-\lambda\Delta t}\frac{(\lambda\Delta t)^k}{k!}$$

Assumption 2: The capital asset pricing model (CAPM) holds for equilibrium security returns and the jump component of a firm's value equation (9) is purely firm-specific and is uncorrelated with the market. According to Merton (1976), there generally does not exist a set of portfolio weights that will eliminate the jump-risk. A Black-Scholes hedge will not be riskless even in a continuous-time setup. However, if the jumps component represents nonsystematic risk, a portfolio which removes the risk of diffusion component (i.e., dW does not appear in the return process of the portfolio) will have a zero "beta". By the CAPM, the expected return on that portfolio must equal the riskless rate. The jump risk will therefore not receive a risk premium.

With an increase in complexity, assumption 2 may be replaced by some alternative assumption. For example, let the considered process differ from some work on option pricing under jump-diffusion processes (Merton (1976), and Ball and Tourous (1983, 1985)) in an important direction. That is, the jumps are allowed to be asymmetric, i.e., with nonzero mean. Values of the expected percentage jump size μ_r greater (less) than zero imply that the distribution is positively (negatively) skewed relative to geometric Brownian motion. This approach is common in the existing literature (e.g., Bates (1996)), and it can be formulated as follows.

Define W as a optimally invested wealth of a representive investor and

$$\Psi(W,t) = \max_{\{C(t)\}} \mathrm{E}\left[\int_t^{\infty} U(C(t),t)dt + \Phi(W(T))\right]$$

as her/his indirect utility of wealth at time t under the optimal consumption plan $C(t)$, which is a solution to the so called Hamilton-Jacobi-Bellman equation.[4] $\Phi(W(T))$ is a bequest utility function in T, which measures the utility of having some money left at the end of the period, and $U(C(t), t)$ denotes a utility function at time t under consumption function $C(t)$.

Having established the stock price dynamics, we now turn to the dynamics of the option price. Suppose that the option price, $P(t)$, can be written as a twice-continuously differentiable function of the stock price and time: namely $P(t) - F(S(t), t)$. According to Bates (1991), if the stock price follows a jump-diffusion process described in (9), then the option price (call or put) F (written as a function of time until expiration instead of time) must satisfy:

$$0 = \frac{1}{2}\sigma(t)^2 S(t)^2 F_{SS} + \left(\mu(t) - \lambda \mathrm{E}\left[\frac{\Psi_W^*}{\Psi_W} J(t)\right]\right) S(t) F_S + F_t - g(S(t),t)F + \\ + \lambda \mathrm{E}\left[\frac{\Psi_W^*}{\Psi_W}\left(F(S(t)J(t),t) - F(S(t),t)\right)\right] \qquad (12)$$

subject to the boundary conditions:

$F(0, t) = 0,$

$F(S(T), T) = \max(0, |S(T)-K|)$

where *g(S(t), T)* denote the equilibrium, instantaneous expected rate of return on the option when the current stock price is *S(t)* and the option expires at time *T* in the future, *K* is the exercise price of the option, Ψ_w is the mariginal utility of dollar wealth of the market-average representive investor, and $\Psi_w^* = \Psi_w(W(1+J(t)),t)$. The indirect utility function $\Psi(W(1+J(t)),t)$ plays the decisive role in risk neutralizing stock's drift and jump terms, and hence in the pricing of jump risks.

As it was shown, equation (12) is a mixed partial differential-difference equation, and although it is linear, such equations are difficult to solve. In order to apply the described valuation option equation (12) one has to follow two steps: (i) solve explicitly for Ψ(W(t)) - the indirect utility function the Hamilton-Jacobi-Bellman equation from the inverstor's consumption-portfolio problem, and (ii) substitute the indirect utility function into the fundamental valuation equation and solve for the option prices. Unfortunately, one typically cannot find a closed-form solution for the indirect utility function, unless the investor has a log period utility or the investment opportunities are non-stochastic. For this reason, virtually all existing equilibrium valuation models for option prices assume two classes of utility functions: the power (particularly the log utility function) and the exponential utility class. Economically, these two classes are interesting because the power utility functions form the constant relative risk aversion class while the exponential utility functions represent the constant absolute risk aversion class. By construction, jump-risk is systematic: all asset prices and wealth jump simultaneously, albeit by possibly diffrent amounts. Bates (1991, 1996) finds that under a traditional assumption of preferences, that is, if the representive investor has a power utility function

$$U(C(t),t) = e^{-\rho t}\left[\frac{C(t)^{\gamma-1}-1}{\gamma-1}\right],$$

where ρ is a constant discount factor, γ is the coefficient of relative risk aversion, which is constant, independent of both wealth and the state variables in this case[5] – then the risk-neutral movement of the underlying asset price follows a jump-diffusion process similar to the one under assumption 1 and the price of option satisfies the following partial differential equation:

$$0 = \frac{1}{2}\sigma(t)^2 S(t)^2 F_{SS} + (r - \lambda^* \mu_J^*)S(t)F_S + F_t - rF + + \lambda^* \mathrm{E}[(F(S(t)J(t)^*,t) - F(S(t),t))] \tag{13}$$

where

r is riskless instantaneous return,

$$\lambda^* = \lambda \mathrm{E}\left[\frac{\Psi_W^*}{\Psi_W}\right],$$

$(1+ J(t)^*)$ is a lognormal random variable: $\mathrm{E}[1 + J(t)^*]=1+ \mu_J^*$, and

$$\mu_J^* = \mu_J + \frac{\operatorname{cov}\left(J, {\left(\Psi_W^* - \Psi_W\right)}\big/{\Psi_W}\right)}{\mathrm{E}\left[1+{\left(\Psi_W^* - \Psi_W\right)}\big/{\Psi_W}\right]}$$

As was shown, the assumption 2 simplifies the model but does not change the main conclusions of this paper.

Pricing European options from the risk-neutral valuation equation (13) is straighforward; see Merton (1976) or Bates (1991). European calls are priced as the expected value of their terminal payoffs under the risk-neutral probability measure are then given by the following closed-form

$$\begin{aligned}\Pi_C(t) &= e^{-r\tau}\sum_{k=0}^{\infty}\Pr{}^*\{k \quad \text{jumps}\}\mathrm{E}^*\left[\max(S_T - K, 0) \mid k \quad \text{jumps}\right] \\ &= e^{-r\tau}\sum_{k=0}^{\infty} e^{-\lambda^*\tau}\frac{(\lambda^*\tau)^k}{k!}\left[S(t)e^{r(k)\tau}N(d_{1k}) - KN(d_{2k})\right]\end{aligned} \tag{14}$$

where

τ = T-t

$$r(k) = r - \lambda^*\mu_J^* + \frac{k\ln\left[1+\mu_J^*\right]}{\tau}$$

$$d_{1k} = \frac{\ln\left[\dfrac{S(t)}{K}\right] + r(k)\tau + \dfrac{1}{2}\left(\sigma^2\tau + k\sigma_J^2\right)}{\sqrt{\sigma^2\tau + k\sigma_J^2}}$$

$$d_{2k} = d_{1k} - \sqrt{\sigma^2 \tau + k\sigma_J^2}$$

European puts have an analogous formula:

$$\begin{aligned}\Pi_P(t) &= e^{-r\tau} \sum_{k=0}^{\infty} \Pr{}^*\{k \quad \text{jumps}\} \mathrm{E}^*[\max(K - S_T, 0) \mid k \quad \text{jumps}] \\ &= e^{-r\tau} \sum_{k=0}^{\infty} e^{-\lambda^* \tau} \frac{(\lambda^* \tau)^k}{k!} [KN(-d_{2k}) - S(t) e^{r(k)\tau} N(-d_{1k})]\end{aligned} \tag{15}$$

Parameters estimation. In applying option pricing formulae (14–15), one has first to determine the parameters: the spot volatility and the structural parameters, which are unobservable. It has been common to apply the described formulae by means of the option-implied parameters based on the model. The interest in implied paramters reflects the fact that options are forward-looking assets, with prices sensitive to distributional moments such as future volatility. A major problem with implied parameter estimation is that there is no associated statistical theory. However, one can in principle apply econometric tools such as maximum likelihood or the generalized methods of moments to obtain the required estimates from time series data on the underlying asset price (e.g., Ball and Torous (1985). However, such estimation may not be practical or convenient, because of its stringent requirement on historical data. To follow this approach here we derive the analytics required for maximum likelihood as well as the analytics for the method of moments estimation.

1. The generalized method of moments. The moments of the jump-diffusion process offer valuable insights. First, the behavior of the options price may be inferred form a study of the moments. Second, the moments are easily used in method of moments estimation models.

Let m_i^J denote the i-th moment of the distribution $\mathrm{N}\left(\ln[1+\mu_J] - \frac{1}{2}\sigma_J^2, \sigma_J^2\right)$. Let us consider the asset return process *Z(t)* of the form (11) as a risk-neutral process.

Lemma 1

In the jump-diffusion process Z(t), the first four moments are given by the following expressions:

$$\mathrm{E}[Z(t)] = [\mu + \lambda m_1^J]\Delta t \tag{16}$$

$$E\left[Z(t)^2\right] = \left[\sigma^2 + \lambda m_2^J\right]\Delta t + \left[\mu + \lambda m_1^J\right]^2 (\Delta t)^2 \tag{17}$$

$$E\left[Z(t)^3\right] = \lambda m_3^J \Delta t + 3\left[\mu + \lambda m_1^J\right]\left[\sigma^2 + \lambda m_2^J\right](\Delta t)^2 + \left[\mu + \lambda m_1^J\right]^3 (\Delta t)^3 \tag{18}$$

$$\begin{aligned} E\left[Z(t)^4\right] = \lambda m_4^J \Delta t + \left[3\left[\sigma^2 + \lambda m_2^J\right]^2 + 4\left(\lambda m_3^J\right)\left[\mu + \lambda m_1^J\right]\right](\Delta t)^2 + \\ + 6\left[\mu + \lambda m_1^J\right]^2\left[\sigma^2 + \lambda m_2^J\right](\Delta t)^3 + \left[\mu + \lambda m_4^J\right]^4 (\Delta t)^4 \end{aligned} \tag{19}$$

Since the proof of this lemma are straightforward but quite technical, detailed, and lengthy, generally they have been omitted in the interests of clarity and are available on request from the author.

Let κ_i denotes the i-th sample cumulant of the distribution of asset returns *Z(t)*. Using the relationships between sample cumulants κ_1, κ_2, κ_3 and κ_4 of a distribution and its moments[6], after some algebra from equations (16 –19), it can be shown that

$$\lambda m = \frac{\kappa_1}{\Delta t} - \mu \tag{20}$$

$$\lambda\left(m^2 + v\right) + \upsilon = \frac{\kappa_2}{\Delta t} 1 \tag{21}$$

$$\lambda\left(m^3 + 3mv\right) = \frac{\kappa_3}{\Delta t} \tag{22}$$

$$\lambda\left(m^4 + 6m^2 v + 3v^2\right) = \frac{\kappa_4}{\Delta t} \tag{23}$$

where

$$m = \ln\left[1 + \mu_J\right] - \frac{1}{2}\sigma_J^2$$

$$v = \sigma_J^2$$

$$\upsilon = \sigma^2 .$$

Since $\kappa_1, \ldots, \kappa_4$ may be determined by observation, therefore setting the sample cumulants equal to the population cumulants yields four algebraic equations in the unknown parameters $(\lambda, \mu_J, v, \upsilon)$. Successive substitutions in (23) readily provide the following equation

$$\frac{36(\kappa_1 - \mu\Delta t)^2}{\kappa_4}\nu^4 - \frac{12\kappa_3^2}{\kappa_4^2}\nu^2 + 3\nu + \frac{\kappa_3^4}{(\kappa_1 - \mu\Delta t)^2\kappa_4^2} - \frac{\kappa_3}{\kappa_1 - \mu\Delta t} = 0 \qquad (24)$$

This equation can be easily solved by using MATHEMATICA – a software product designed by Wolfram Research Inc. The correct root of equation (24) ν could then be used to obtain three other estimates μ_J, λ and υ by substituting it into equations (20-23). It may be shown that

$$\mu_J = \exp\left[\frac{\kappa_3^2}{(\kappa_1 - \mu\Delta t)\kappa_4 - 6\left(\left(\kappa_1 - \mu\Delta t\right)/\kappa_4\right)\nu^2 + \nu/2}\right] - 1 \qquad (25)$$

$$\lambda = \frac{\kappa_4/\Delta t}{\kappa_3^2/(\kappa_1 - \mu\Delta t)^2 - 6\nu^2} \qquad (26)$$

$$\upsilon = \frac{\kappa_2}{\Delta t} - \frac{\kappa_4/\Delta t}{\kappa_3^2/(\kappa_1 - \mu\Delta t)^2 - 6\nu^2}\left(\frac{\kappa_3}{\kappa_1 - \mu\Delta t} - 2\nu\right) \qquad (27)$$

Press (1967) estimated jump-diffusion processes with zero instantaneous expected rate of return, $\mu = 0$ for a sample of monthly returns to ten NYSE listed common stocks over the period 1926 through 1960, using the method of cumulants, however he frequently obtained negative estimates of the variance parameters υ and ν. Becker (1981) similarly estimated jump-diffusion processes for 47 NYSE listed common stocks over the period 15 September, 1975 through 7 September, 1977, and in opposition to Press's work, he set the mean logarithmic jump size equal to zero, $\mu_J = 0$ and also often obtained negative estimates of the variance parameters υ and ν. Note that a negative estimate of variance is strictly the result of using the generalized method of moments rather than being caused by a model specification error.

2. The maximum likelihood estimation procedure. Parameter estimation by the generalized method of moments is known to yield consistent estimators. However, these estimators are not always efficient but it may often be possible, because of the large quantities of market data avaiable, to ignore efficiency and rely upon consistency. Because the "moment" approach often provides negative estimates of the variance parameters, very few papers have employed this method to estimate the return asset processes as of current writing. Most of papers dealing with the jump-diffusion processes have used maximum likelihood estimation along with a truncation of the infinite series representation of the likelihood function.

The estimation of a jump-diffusion process by means of the maximum likelihood method requires the probability density of the analyzed process. This section provides the following result to its characteristic function and the density function stated in the form of a lemma for ease of future reference.

Lemma 2

Assume that the return asset process follows the jump-diffusion process described as in (11).

Let φ(u) denote its characteristic function, then

$$\ln[\varphi(u)] = \Delta t\left(\mu - \frac{1}{2}\sigma^2 - \lambda\mu_J\right)ui - \frac{1}{2}\sigma^2\Delta t u^2 + \lambda\left(e^{mui - vu/2} - 1\right)\Delta t \tag{28}$$

Let $f(\xi)$ denote the density function, then

$$f(\xi) = \sum_{k=0}^{\infty}\frac{e^{-\lambda\Delta t}(\lambda\Delta t)^k}{k!}\eta\left(\xi;\left(\mu - \frac{1}{2}\sigma^2 - \lambda\mu_J\right)\Delta t + km, \upsilon\Delta t + kv\right) \tag{29}$$

where

$$m = \ln[1 + \mu_J] - \frac{1}{2}\sigma_J^2$$

$$v = \sigma_J^2$$

$$\upsilon = \sigma^2$$

$$\eta(\xi; a, b) = \frac{1}{\sqrt{2\pi b}}e^{-(\xi - a)^2/2b}, \textit{ and}$$

$$i = \sqrt{-1}$$

Proof. The proof is avaiable on request from the author. The estimation of parameters of the process is carried out by maximum likelihood, using discrete time series of asset returns $\mathbf{Z} = (Z(t),\ t = 0, 1, \ldots, T)$ as data and denote $\Omega = (\lambda, \mu_J, \nu, \upsilon)$, then the estimator $\overline{\mathbf{\Omega}}$ of those parameters is obtained by maximizing the log likehood function:

$$\max_{\{Z,\Omega\}}\left\{\mathbf{L}(\mathbf{Z};\mathbf{\Omega})=\sum_t L_t(\mathbf{Z}(t);\mathbf{\Omega})=\sum_{t=0}^{T-1}\ln[f(\mathbf{Z}(t);\mathbf{\Omega})]\right\} \tag{30}$$

where $f(Z(t);\mathbf{\Omega})$ is given by equation (30).

Estimation and testing can simply be carried out in terms of these derivatives. The first derivative $\partial\mathbf{L}/\partial\mathbf{\Omega}$ can be written compactly in terms of the T x 4 array II with the element

$$\mathbf{\Pi}=[\pi_{ti}]_{T\times 4}=\frac{\partial L_t}{\partial\Omega_i}$$

The necessary conditions for the existence of maximum likelihood estimators $\overline{\mathbf{\Omega}}$ is provided by

$$\frac{\partial\mathbf{L}(\mathbf{Z};\overline{\mathbf{\Omega}})}{\partial\Omega_i}=0. \qquad i=1,2,3,4$$

whereas the corresponding sufficient conditions constitute the positive definiteness of $-\ \mathbf{H}(\mathbf{Z};\overline{\mathbf{\Omega}})$, the Hessian matrix $\mathbf{H}_{4\times4}$ being defined by

$$\mathbf{H}=[H]_{ij}=\frac{\partial^2\mathbf{L}(\mathbf{Z};\mathbf{\Omega})}{\partial\Omega_i\partial\Omega_j}$$

The derivatives are computed numerically, and a ready solution to the maximization of this likelihood function is to adopt the Berndt, Hall, Hall and Haussman (1974) approach using the iteration

$$\mathbf{\Omega}^{k+1}=\mathbf{\Omega}^{k}+\varsigma\mathbf{H}^{-1}\mathbf{\Pi}'\mathbf{I}=\mathbf{\Omega}^{k}+\varsigma(\mathbf{\Pi}'\mathbf{\Pi})^{-1}\mathbf{\Pi}'\mathbf{I}$$

with ς as a step length which is adjusted from its a priori value of unity by a simple line search, II' is as the transpose matrix of the first derivatives evalued at Ω^{κ}, and I is a $T\times 1$ unit vector so the first order condition is satisfied, that is simply $\Pi'I = 0$.

A GARCH model

This sections presents alternative methods for pricing options. Empirical evidence on underlying asset prices and on their derivatives strongly suggests that asset price volatility is usually unstable through time. Black (1976), Christie (1982), among others, discovered an inverse correlation between asset returns and changes in volatility. This peculiar feature of asset return suggests that the asset price volatility itself should be modelled by means of a stochastic process. The pricing of options under this condition is, therefore, an important problem. Typically, in a continuous-time framework, volatility $\sigma(t)$ is assumed to follow a diffusion process. Let the asset price S be given by the expression

$$dS(t) = \mu(S(t), t)dt + \sigma(t)S(t)\,dW \tag{31}$$

with the volatility σ satisfying

$$d\sigma(t) = \mu_\sigma(\sigma(t), t)\,dt + v dW_\sigma \tag{32}$$

then the price function $P(t) = F(s,\sigma,t)$ of a European option can be shown to satisfy the following partial differention equation (cf. Hull and White, 1976)

$$F_t + \frac{1}{2}\sigma^2 s^2 F_{ss} + rsF_s - rF + \frac{1}{2}v^2 F_{\sigma\sigma} + (\mu_\sigma + \lambda v)F_\sigma = 0 \tag{33}$$

where $\lambda(\sigma, t)$ represents the market price for volatility risk, which needs to be exogenously specified. For some specifications of the dynamics of stochastic volatility and the market price for risk, a closed-form expression for the option's price is available, otherwise, numerical procedures need to be employed. Since this approach is beyond the scope of this article, for more details we refer to Hull and White (1987, 1988), Johson and Shanno (1987), Stein and Stein (1991), Heston (1993), Ball and Rome (1994), Bakshi et al. (1996), among others.

Still another approach to the modelling of stochastic volatility is formulated in a discrete-time framework based in so-called *general autoregressive conditional heteroskedasticity* – GARCH. We say very little about (G)ARCH models because several excellent surveys on the subject have appeared for some time including those by Engle and Bollerslev (1986), Bollerslev, Chou and Kroner (1992), Bolerslev, Engle and Nelson (1994).

There are many different types of ARCH models that have a wide variety of applications in finance. The two most popular ARCH process are generalized ARCH (GARCH) (Bollerslev (1986)) and exponential GARCH (EGARCH (Nelson (1991)). The technical distinctions are beyond the scope of this article; however, researchers have tended mostly to use the GARCH process and its variations for option pricing. We now review the basic GARCH option pricing model of Duan (1995).

Let S_t be the asset price at date t, and $v(t)$ be the conditional variance, given information at date t, of the logarithmic return over the period $[t, t+1]$ which (without loss of generality) we call a "day". The dynamics of prices are assumed to follow the process

$$\ln\left[\frac{S_{t+1}}{S_t}\right] = r_f + \lambda\sqrt{v_{t+1}} - \frac{1}{2}v_{t+1} + \sqrt{v_{t+1}}\varepsilon_{t+1}, \text{ for } t = 0, 1, 2, \ldots \quad (34)$$

$$v_{t+1} = \beta_0 + \beta_1 v_t + \beta_2 v_t(\varepsilon_t - \theta)^2, \text{ for } t = 0, 1, 2, \ldots \quad (35)$$

$$\varepsilon_t \xrightarrow{P} N(0,1) \quad (36)$$

where r_f is the one-period, continuously compounded return on the risk-free asset, λ is a constant unit risk premium, v_{t+1} is the conditional variance of the asset return, and [ε_t: t = 0, 1, 2, ...] form a sequence of independent standard normal random variables with respect to measure P. Equations (34 – 36) state that conditional on time t information, the price at time $t+1$ is a drawing from a log-normal distribution with

$$\mathrm{E}_t(S_{t+1}) = S_t e^{r_f + \lambda\sqrt{v_t}}$$

$$\mathrm{var}_t(S_{t+1}) = \mathrm{E}_t\left[S_{t+1}^2\right]\left[e^{v_t} - 1\right]$$

The conditional variance v_{t+1} for the period [$t+1$, $t+2$] depends on its level in the previous period, v_t, and on the standard normal innovation over the time period [t, $t+1$].

The particular structure imposed in equation (34) is the nonlinear asymmetric GARCH (NGARCH) process with the typical GARCH parameter restrictions: $\beta_0 > 0, \beta_1 > 0, \beta_2 > 0$, that has been studied by Engle and Ng (1993), and Duan (1995). Parameter q determines the "leverage effect", the nonnegative θ captures the negative correlation between asset return and conditional volatility innovations that is frequently observed in stock markets.[7] Duan (1995) has estabilished via an equilibrium argument that the underlying asset price dynamics under the locally risk-neutralized probability measure Q can be written as

$$\ln\left[\frac{S_{t+1}}{S_t}\right] = r_f - \frac{1}{2}v_{t+1} + \sqrt{v_{t+1}}\varepsilon_{t+1}, \text{ for } t = 0, 1, 2, \ldots \quad (37)$$

$$v_{t+1} = \beta_0 + \beta_1 v_t + \beta_2 v_t(\varepsilon_t - \theta - \lambda)^2, \text{ for } t = 0, 1, 2, \ldots \quad (38)$$

$$\varepsilon_t \xrightarrow{Q} N(0,1) \quad (39)$$

where the residuals obtained from the process $\varepsilon_t = \varepsilon + \lambda$ is a standard normal random variable under the locally risk-neutralized probability measure Q. This risk-neutralized system serves as the backbone for option pricing under which option prices can be computed as a simple discounted expected value of its payoff value. Duan has presented an analytical approximation for a Eurepean call option price with strike price K and maturity T in the form

$$\Pi_{C_{t,pp}} = \Pi_C + \kappa_3 A_3 + (\kappa_4 - 3) A_4 \tag{40}$$

where

$$\Pi_C = S_0 N(\bar{d}) - Ke^{-rT} N(\bar{d} - \sigma_{\tau T}) \tag{41}$$

A_3 and A_4 are defined as

$$A_3 = \frac{1}{3!} S_0 \sigma_{\tau T} \left[\left(2\sigma_{\tau T} - \bar{d} \right) n(\bar{d}) - \sigma_{\tau T}^2 N(\bar{d}) \right] \tag{42}$$

$$A_4 = \frac{1}{4!} \left[\left(\bar{d}^2 - 1 - 3\sigma_{\tau T} \left(\bar{d} - \sigma_{\tau T} \right) \right) n\left(\bar{d} \right) + \sigma_{\tau T}^3 \right] \tag{43}$$

$$\bar{d} = d + \delta \tag{44}$$

$$d = \frac{\ln\left[\frac{S_0}{K} \right] + \left(rT + \frac{1}{2} \sigma_{\tau T}^2 \right)}{\sigma_{\tau T}} \tag{45}$$

$$\delta = \frac{\mu_{\tau T} - rT + \sigma_{\tau T}^2 / 2}{\sigma_{\tau T}} \tag{46}$$

with $\tau_T = \ln \frac{S_T}{S_0}$ representing the temporally aggregated asset return, $\mu_{\tau T}$ and $\sigma_{\tau T}$ are the mean and standard devaition of τ_T, conditional on $\mathbf{F}_0$ with respect to the locally risk-neutralized measure Q; κ_3, κ_4 are the skewness and kurtosis coefficient conditional on $\mathbf{F}_0$ under the measure probability Q. The functions $n(\bar{d})$ and $N(\bar{d})$ are, respectively, the density and cumulative distribution functions of the standard normal random variable.

The above model has five unknown parameters, namely $\beta_0, \beta_1, \beta_2$ the sum $(\lambda + \theta)$, and the initial variance v_0. In order to apply the formula (46), we first need to estimate the GARCH parameters $\Omega = [\beta_0, \beta_1, \beta_2, \lambda + \theta]$, then to derive analytical expressions for the four moments of the temporally aggregated returns.

The estimation of parameters of the process is carried out by maximum likelihood, using daily series of underlying asset returns $\mathbf{Z} = (Z(t),\ t = 0, 1, \ldots, T)$ as data and the BHHH algorithm (Berndt, Hall, Hall, Hausman (1974)), given the return series and initial values of ε_0 and an appropriate v_0[8].The log-likelihood function we have to maximize for the process under consideration is the following

$$\mathbf{L}(\mathbf{Z};\boldsymbol{\Omega}) = -\frac{1}{2}T\ln[2\pi] + \sum \ln\left[\frac{1}{\sqrt{v_t}}\right]e^{-\varepsilon_t^2/2v_t} \qquad (47)$$

Data and estimation results

Description of the data

The data. The focus of this study is on stock index options, particular on DAX Index options. The DAX index option contract (DAX) is traded on the Deutsche Terminböse and is by far the most actively traded index option contract in the European continent. Data on DAX Index option prices from September 16, 1991 through April 30, 1997 were avaiable from Deutsche Terminböse. Data on the DAX stock index for this period were hand-collected from *The Financial Times Journal*. The considered index is value weighted. A complication we faced is that the described data set is the daily closing value levels, and the DAX index is not adjusted for dividends. This is not ideal from a theoretical point of view since the use of closing prices is subject to the criticism of nonsynchronous trading. This problem arises when there is a failure to observe the option price and the price of the underlying index simultaneously. However, since all of our strike prices are in the neighbourhood of at-the-money options, which are the most heavily traded, we would expect that the dividend adjustment and the nonsynchronous price problem would not greatly affect our results.

Several exclusion filters are applied to construct the option price data. First, options with less than 7 days to expiration may induce liquidity-related biases, and these are included from the sample. Second, to mitigate the impact of price discreteness on option valuation, option prices lower than $\$\frac{3}{8}$ are not included. Then the option data, call and put, is each divided into several categories according to either moneyness and term to expiration. Let S denote the DAX index level of given day, and K is the exercise price. A call option is then to be deep out-of-the-money (D-OTM) if its $S/K < 0.94$; out-of-the-money (OTM) if $S/K \in [0.94, 0.97)$; slightly at-the-money (S-ATM) if $S/K \in [0.97, 1.0)$; at-the-money (ATM) if

$S/K \in [1.0, 1.3)$; in-the-money (ITM) if $S/K \in [1.3, 1.6)$; and deep in-the-money (D-ITM) if $S/K \geq 1.6$. By the term of expiration, an option contract can be classified as very short-term (< 30 days); short-term (30-60 days); medium-term (60-90 days); slightly long-term (90-180 days); long-term(180-270 days); and very long-term($\geq$ 270 days). The proposed moneyness and maturity classfications produce 36 categories of call options for which the empirical result will be reported.

Descriptive Statistics. The daily returns of DAX Index under consideration are continuously compounded returns. They are calculated as the difference in natural logarithm of the closing index value for two consecutive trading days. The total number of returns for the considered period is 1430. The descriptive statistics for the data are in Table 1.

It can be observed that the distribution is negatively skewed, indicating that it is non-symmetric. Futhermore, it exhibits high levels of kurtosis, which indicates that the distribution has fatter tails than a normal distribution. The Kolmogorov-Smirnov D-Statistics to test the null hypothesis of normality has also been calculated, and it rejects the normality assumption at the significance level of one per cent. The results confirm the well known fact that daily stock returns are not normally distributed, but are leptokurtic and skewed.

Table 1. Descriptive Statistics (DAX Stock Index over the period 19 Septermber 1991 to 30 April 1997. The data is daily in frequency)

Sample size	1430
Mean	0.00051197
Standard deviation	0.01014582
Skewness	-0.64333599
Kurtosis	12.0844185
Komogorov-Smirnov D-statistics	2.47456923
The critical value for test of normality	$l_{0.01}=1.62$

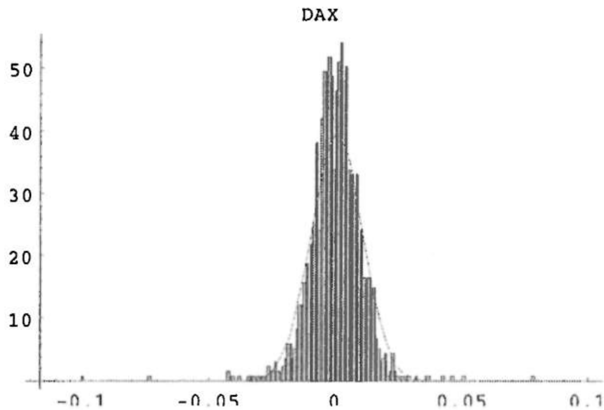

Figure 1. The histogram of the distribution of DAX Index returns over the period 1991–1997

Figure 1 shows the histogram of the empirical distribution of DAX Index returns over the period under consideration. It can be observed from this graph that the empirical distribution has heavy tails and sharp peaks at the center compared with a normal distribution.

The implied risk-free rate of interest. One difficulty we have faced in providing this research is the risk-free rate of interest. If avaiable, one may assume that the risk-free rate of interest is the continuously compounded equivalent yield on United States Treasury bills having the maturity closest to the expiration date of the option. The discount rates on T-bills could be collected from some finance journals, and the risk-free rates are based on the average of the bid and ask discounts. In this paper, we have employed the so-called implied risk-free rate of interest, which is derived from the put-call parity

$$\Pi_C - \Pi_P = \begin{cases} S - \delta - Ke^{-r(T-t)}, & \textit{in case of discrete dividend} \\ Se^{-q(T-t)} - Ke^{-r(T-t)}, & \textit{in case of continuous dividend} \end{cases} \quad (48)$$

where $\Pi_C(S, t)$, $\Pi_p(S, t)$ is the call and put option price at the current time t respectively. S, K, T is the underlying asset price, the strike price, and the time of maturity respectively. If the asset pays discrete dividend d, the implied risk-free rate of interest r is then given by

$$r = -\frac{1}{T-t}\ln\left[\frac{S - \delta + \Pi_P - \Pi_C}{K}\right] \quad (49)$$

If the asset pays known continuous dividend yield q, the implied risk-free rate of interest is as follows

$$r = -\frac{1}{T-t}\ln\left[\frac{Se^{-q(T-t)} + \Pi_P - \Pi_C}{K}\right] \tag{50}$$

If the dividend (δ or θ) is truly known and the prices of a pair of identical options are given, one can estimate the implied risk-free rate of interest in either formula (49) or (50). To compute the interest rate, we ignore the effect of dividend and assume that the index pays no dividend. For a given day and given pairs of call and put options with the same maturity, we estimated the implied interest rate for these options using the above formulas. Then, we equally weigh the implied volatilities of all options avaiable in any given day (and given month) to produce an average daily (and monthly) implied interest rate. Figure 2 presents the montly average of the implied risk-free rates of interest over the period 1991-1997. The risk-free rate of interest used in this paper is set equal to the average value of the montly average implied interest rates, and it is given as 5.32% per year. As we can see, this rate is quite reasonable and is not much different from the one collected from US T-bills.

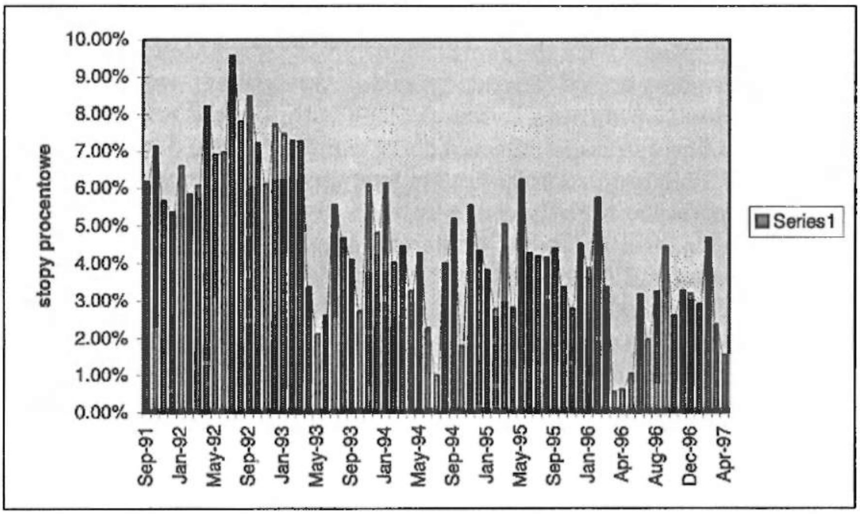

Figure 2. The implied risk-free rate of interest rate on DTB over the period 1991–1997

The implied dividend and implied volatility. To complete the survey on DAX Stock Index returns, let us use the described data set to examine the dividend paid by DAX and the extent and the direction of biases associated with the Black-Scholes model in terms of volatility. Since index dividend is not available in any form, therefore we employ the joint estimation procedure of implied volatility and implied dividend discussed by Choi and Wohar (1992) to solve for the implied volatilities and the implied dividends for the index options from the dividend-adjusted Black-Scholes option pricing formula. The dividend-adjusted version of Black-Scholes pricing formulas are given as follows

$$\Pi_C = [S - \delta]N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{51}$$

$$\Pi_P = Ke^{-r(T-t)}N(-d_2) - [S - \delta]N(-d_1) \tag{52}$$

where

$$d_1 = \frac{\ln\left(\frac{S-\delta}{K}\right) + \left(r + \frac{1}{2}\sigma_t^2(T-t)\right)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

The estimation procedure of implied volatility and dividend is as follows, having calculated implied volatilities and implied dividends for each of three given pairs of call and put options with the same maturity but with different strike prices (slightly in-the-money, out-of-the-money, and at-the-money), we then average the implied dividends and combine this average with the call and put option formulas (51 - 52) to find the three different corresponding implied volatilities. In the next step, these three implied volatilies are averaged and combined with these call and put formulas to find the three implied dividends. The procedure is repeated until both average implied volatility and average implied dividends converge for all option with different expiration terms for each day in our sample. Finally, the implied dividends over the options, all having different exercise prices but the same expiration date are averaged to gain final the present value of implied daily dividend of a given day. The present value of implied daily dividend of given day is then used to find the final implied volatilities of options, all having different exercise prices but the same expiration date.

The above presented method of calculating volatility may be viewed as an average of the expected daily return variances for the underlying index over the time period until expiration. Table 2 presents the average of the present value of implied dividends for different maturity of options for the period Septermber 19, 1991 through April 30, 1997: the average of absolute present value of implied dividends and the ratio of the average annualized implied dividend to the average DAX index. The six-year average DAX index is about 2089.7092. The average yearly dividend yields for DAX index long-term options are 3.32% and 1.59%, respectively. Our results confirm the fact observed in respect of New York Stock Exchange

Composite Index and Standard & Poor's 500 by Choir and Wohar in their work, that is while the average of the estimated long-term implied dividends appear to be reasonable, the short-term and mid-term option on the average seems to overestimate the true dividend yield, which may indicate that there are other factors to be considered in determining short-term and medium-term option prices not included in the Black-Scholes option pricing model.

Table 2. The average of present value of implied dividends (PVD) DAX Stock Index over the period 19 September 1991 through 30 April 1997

The average of:	Days-to-Expiration			
	< 30	30–90	90–180	> 180
absolute PVD	8.419278	11.972860	12.414370	16.699410
percentage PVD	4.0328%	4.5835%	3.3203%	1.5980%

Table 3A and table 3B report the average Black-Scholes implied volatility values across five moneyness and four maturity categories, for calls and puts respectively as well as for both the entire sample period and different subperiods. The sample period extends from September 1991 through April 1997 for a total of 38,427 calls and 40,606 puts. As was shown, regardless of the sample period and term to expiration, the implied volatility exhibits a strong U-shaped pattern as the call options go from D-OTM to ATM and then to D-ITM or as the put options go from D-ITM to ATM and then to D-OTM, with the deepest ITM call-implied and the deepest OTM put-implied volatility taking the highest values. Practioners refer to this phenomenon as the volatility smile, where the so-called "smile" refers to the U-shaped pattern of implied volatilities accross different strike prices. Futhermore, regardless of option type, the volatility smiles are strongest for short-term options (reg, indicating that short-term options are the most severely mispriced by the Black-Scholes model). For a given sample period and moneyness range, the volatility smile is downward-sloping in most cases and exhibits a slight U-shape in some cases, as the term to expiration increases. These findings of clear moneyness-related and maturity-related biases asscociate with the Black-Scholes are consistent with those in the existing literature (see for instance Rubinstein (1985), Taylor and Xu (1993), Bates (1996)...).

It is widely belived that volatility smiles have to be explained by alternative models, for instance a model with stochastic volatility or model jump-diffusion. As the smile evidence is indicative of negatively-skewed empirical distribution returns with excess kurtosis, an alternative model must be based on a distributional assumption that allows for negative skewness and excess kurtosis. However, other arguments to explain the smile and its skewness (jump transaction costs, bid-ask spreads, non-synchronous trading, liquidity problems ...) have also to be taken in account for both theoretical and empirical reasons. For instance, there exists empirical evidence suggesting that the most expensive options (the upper parts of the smile curve) are also the least liquid; skewness may therefore be attributed to specific configurations of liquidity in option markets.

Note that the implied volatility of calls in a given category is similar to the implied volatility of puts in the opposing category, regardless of sample period or term to expiration. Such similarties in pricing structures exist between calls and puts mainly due to the working of put-call parity. For this reason, basing the discussions to follow solely on results obtained from DAX calls should not present a biased picture of the empirical study.

Jump-diffusion parameter and N-GARCH (1,1) parameter estimation results

The jump-diffusion parameters for each day of the period from 01 January, 1997 through 30 April, 1997 were estimated using the maximium likelihood methodology. The log-likelihood function in equation (30) is used for the estimation.

Figure 3 presents the probability density function of $\ln\frac{S_{t+1}}{S_t}$ over the period from 01 January, 1997 through 30 April, 1997. From the graph it is known that the period under consideration was quite quiet. The N-GARCH (1,1) parameters for all days of the period from 01 January, 1997 through 30 April, 1997 were also estimated using the maximum likelihood method.

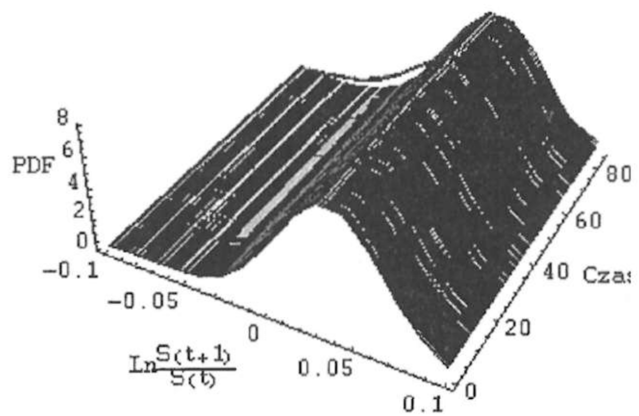

Figure 3. Probability density function of $\ln\frac{S_{t+1}}{S_t}$ over the period from 01 January, 1997 through 30 April, 1997

Table 3A. The average of implied volatility of call options

Sample period	Moneyness		Days-to-Expiration				Subtotal
	S/K		< 30	30-90	90-180	> 180	
09-1991 12-1992	D-OTM	< 0.94	18.3778	15.2745	15.5236	16.0120	1168
	OTM	0.94-0.97	13.1107	13.6433	15.6023	17.2806	1474
	ATM	0.97-1.03	12.8510	15.5127	17.5279	17.9546	4095
	ITM	1.03-1.06	18.6378	17.5460	18.1745	18.8697	906
	D-ITM	> 1.06	27.1001	20.8704	20.9728	20.8332	480
Subtotal			1656	3851	1868	748	*8123*
01-1993 12-1993	D-OTM	< 0.94	14.9665	13.0217	13.7544	14.3017	170
	OTM	0.94-0.97	12.7058	13.3906	14.2439	15.6243	1004
	ATM	0.97-1.03	12.4557	14.9128	15.7716	15.6193	3735
	ITM	1.03-1.06	19.5215	15.7938	16.0636	15.9880	1182
	D-ITM	> 1.06	30.5859	20.6710	17.6514	16.3828	1449
Subotal			1346	3487	1814	893	*7540*
01-1994 12-1994	D-OTM	< 0.94	18.4924	17.6581	17.4881	17.3234	804
	OTM	0.94-0.97	15.7915	17.4476	18.1439	18.1417	1260
	ATM	0.97-1.03	15.8776	19.0631	19.5158	18.1960	3645
	ITM	1.03-1.06	21.4101	19.1520	18.8840	18.6301	878
	D-ITM	> 1.06	30.2682	22.5731	19.7614	17.7853	769
Subtotal			1807	3654	1479	446	*7386*
01-1995 12-1995	D-OTM	< 0.94	16.7070	13.6754	13.5885	13.6581	582
	OTM	0.94-0.97	12.4109	12.7742	13.4934	14.4316	1263
	ATM	0.97-1.03	12.3013	13.6403	14.1815	14.0703	3982
	ITM	1.03-1.06	18.6035	13.8730	13.3868	13.7317	956
	D-ITM	> 1.06	29.2705	16.9996	14.3688	13.8093	517
Subtotal			1620	3619	1423	638	*7300*
01-1996 12-1996	D-OTM	< 0.94	13.2682	10.6509	11.1428	11.1925	471
	OTM	0.94-0.97	10.1735	10.2411	11.0578	11.7940	1132
	ATM	0.97-1.03	10.3191	11.2143	11.8176	11.4541	4434
	ITM	1.03-1.06	19.1087	11.9110	11.4506	10.9138	1187
	D-ITM	> 1.06	32.2043	16.6571	12.3966	9.9096	854
Subtotal			1732	3910	1519	917	*8078*
09-1991 12-1996	D-OTM	< 0.94	17.3663	14.4450	14.8138	13.8804	3195
	OTM	0.94-0.97	12.9638	13.4953	14.7542	15.4828	6133
	ATM	0.97-1.03	12.7749	14.9174	15.8428	15.4127	19921
	ITM	1.03-1.06	19.3497	15.7286	15.4138	15.0267	5109
	D-ITM	> 1.06	30.0363	19.7674	17.0789	15.4142	4069
Subtotal			8161	18521	8103	3642	*38427*

Table 3B. The average of implied volatility of put options

Sample period	Moneyness		Days-to-Expiration				Subtotal
	K/S		< 30	30 - 90	90 - 180	> 180	
09-1991	D-OTM	< 0.94	21.1242	17.3121	15.5373	15.4396	514
12 -1992	OTM	0.94-0.97	15.4592	14.9862	13.9506	13.6814	963
	ATM	0.97-1.03	12.3398	13.1945	13.4133	13.2185	4060
	ITM	1.03-1.06	15.9127	11.3548	11.2305	12.0241	1225
	D-ITM	> 1.06	28.7801	13.2565	10.2431	10.2830	502
Subtotal			1508	3440	1618	698	*7264*
01-1993	D-OTM	< 0.94	19.9471	18.1278	17.5940	17.2031	2016
12-1993	OTM	0.94-0.97	15.1137	15.9530	15.9032	15.8790	1363
	ATM	0.97-1.03	13.0527	15.0049	15.2531	15.2373	3771
	ITM	1.03-1.06	16.5483	13.7655	13.5494	15.0131	922
	D-ITM	> 1.06	24.3320	15.2997	12.7240	13.1883	119
Subtotal			1616	3759	1905	911	*8191*
01-1994	D-OTM	< 0.94	22.3575	22.3177	21.7894	20.7475	1064
12-1994	OTM	0.94-0.97	18.8928	20.6777	20.7616	20.5873	997
	ATM	0.97-1.03	17.0133	20.3906	20.8153	20.1593	3696
	ITM	1.03-1.06	20.8699	18.9154	19.3491	20.3946	1167
	D-ITM	> 1.06	30.5593	20.7801	19.3984	17.7853	694
Subtotal			1908	3757	1507	446	*7618*
01-1995	D-OTM	< 0.94	18.4048	17.0587	16.7857	17.2068	934
12-1995	OTM	0.94-0.97	14.6966	15.5858	16.0529	16.7132	1186
	ATM	0.97-1.03	12.8115	15.3327	16.5042	16.7667	4016
	ITM	1.03-1.06	18.0937	14.7234	15.6158	17.3403	1140
	D-ITM	> 1.06	26.9899	18.4912	16.6102	17.0525	480
Subtotal			1750	3848	1502	656	*7756*
01-1996	D-OTM	< 0.94	19.2950	16.9383	17.4112	18.0205	2198
12-1996	OTM	0.94-0.97	13.9735	14.6577	15.9545	16.9107	1643
	ATM	0.97-1.03	11.4319	13.7513	15.5459	16.5180	4506
	ITM	1.03-1.06	16.8674	16.6771	14.6736	17.0643	1006
	D-ITM	> 1.06	19.2626	16.6865	16.3162	17.3346	424
Subtotal			2137	4711	1847	1082	*9777*
09-1991	D-OTM	< 0.94	20.1433	18.2545	17.8301	17.5453	6726
12-1996	OTM	0.94-0.97	15.5895	16.2446	16.4119	16.2535	6152
	ATM	0.97-1.03	13.2761	15.3978	16.1745	16.0988	20049
	ITM	1.03-1.06	17.8608	14.3785	14.6507	15.6822	5460
	D-ITM	> 1.06	28.1228	17.5942	15.2581	16.0717	2219
Subtotal			8919	19515	8379	3793	*40606*

The performance of jump-diffusion and N-GARCH pricing options models

In this section, we examine each model's out-of-sample cross-sectional pricing performance. For this purpose, we rely on the time series of the index levels and use them as input to compute current day's parameters of the considered processes. Next, we subtract the model-determined price from its observed counterpart, to compute both the absolute pricing error and the percentage pricing error. This procedure is repeated for every call and each day in the sample period from 01 January, 1997 through 30 April, 1997. Table 4 and 5 report the pricing results for the jump-diffusion model pricing option and N-GARCH model pricing option respectively, where for clarity the standard errors for each estimate are omitted as they are generally very small and close to zero. 36 groups of results under the proposed moneyness and maturity classifications are presented to reflect differences in the models' price calculations. For a given model, we compute the price of each option using the previous day's parameters. The average of market price (AP), the average of the absolute pricing errors (AE), the average of the absolute percentage pricing errors (APE) and the average of the percentage of pricing errors (PE) are reported in the tables. The absolute pricing error is the sample average of the model price minus the market price, divided by the market price. The absolute percentage pricing error is the sample average of the absolute perentage of pricing errors. The percentage pricing error is the sample average of the percentage of pricing errors.

According to the three above measures, first, the jump-diffusion model generally does better than the N-GARCH model, except that for a few categories the N-GARCH performs slightly better than the frist one. As was shown, the jump-diffusion model outperforms the N-GARCH model in pricing the deep-out-of-the-money, out-of-the-money and slightly at-the-money options, regardless of maturity. For example, take a deep-out-of the-money call with moneyness less than 0.94 and with 60-90 days to expiration. From tables 4 and 5, the average price for such a call is DM 10.09. When the N-GARCH model is applied to value this call, the resulting absolute pricing error is, on average, DM 6.02 as shown in table 5, but when the jump-diffusion model is applied, the average error goes down to DM 2.32. As another example, for slightly at-the-money calls with short-term-to-expiration (30-60 days), their average price is DM 33.48, the N-GARCH model gives an average pricing error of DM 6.04, and the jump-diffusion model results in an average error of DM 3.50. However, the N-GARCH model performs slightly better in pricing at-the-money, in-the-money and some cases of deep in-the-money options. Regardless of maturity, incorporating N-GARCH model produces by far the most imporant improvement over the jump-diffusion model, reducing the absolute percentage errors typically by 4 percent to 6 percent. Second, observe that two models produce negative percentage pricing errors for in-the-money options with short-term and medium-term of maturity, subject to their time-to-expiration not exceeding 180 days. This means that both models tend to underprice in-the-money options with short term and medium term of maturity while they overprice out-of-the-money options. From the results it can be shown that in-the-money options with very long term of maturity are better valued with the N-GARCH model. Third, generally for a given moneyness category and regardless of the pricing model, the absolute and the absolute percentage pricing errors typically increase from short- to medium- to long-term options. But for a given term of maturity, both models price in-the-money more accurately than out-of-the-money options. And the longer the term of maturity, the more accurate are the model-determined option prices.

Table 4. Out-of-sample pricing errors from a jump-diffusion option pricing model

Moneyness (S/K)		Days-to-maturity					
		6-30	30 - 60	60-90	90-180	180-270	> 270
< 0.94	AP	1.25	4.92	10.09	22.22	45.07	89.82
	AE	0.53	1.24	2.32	5.12	12.02	10.46
	APE	66.39%	36.62%	36.72%	21.99%	25.29%	29.46%
	PE	1.67%	9.51%	14.81%	5.44%	19.18%	21.16%
0.94 - 0.97	AP	3.72	12.94	24.87	47.24	80.72	152.81
	AE	2.02	3.78	5.24	2.71	10.33	55.4
	APE	56.69%	31.75%	22.04%	14.47%	15.79%	29.13%
	PE	35.91%	19.35%	11.86%	0.21%	13.38%	16.93%
0.97-1.00	AP	14.03	33.48	50.23	76.27	111.71	201.98
	AE	3.28	6.50	7.17	17.98	35.19	71.52
	APE	57.44%	18.88%	16.58%	16.08%	15.34%	26.27%
	PE	27.07%	0.43%	4.23%	11.13%	2.30%	9.35%
1.00 - 1.03	AP	49.67	68.97	85.11	112.08	146.23	261.25
	AE	-6.92	-10.77	-14.65	-23.81	62.10	55.06
	APE	17.22%	13.61%	13.66%	15.27%	16.78%	22.94%
	PE	-3.26%	-7.62%	-9.39%	-12.40%	0.21%	6.43%
1.03 - 1.06	AP	104.87	116.31	129.43	156.55	189.79	312.19
	AE	-18.55	-16.24	-19.63	-25.98	62.68	49.12
	APE	11.51%	9.86%	11.57%	12.11%	12.77%	20.48%
	AE	-10.03%	-7.96%	-8.16%	-9.38%	(7.09%)	-14.39%

> 1.06	AP	256.05	239.94	257.55	262.44	273.28	442.83
	AE	-26.10	-32.13	-35.26	-45.49	79.44	127.21
	APE	7.64%	9.43%	9.65%	9.51%	9.91%	16.78%
	AE	-6.82%	-8.57%	-9.17%	-10.15%	-17.81%	(14.09%)

Table 5. Out-of-sample pricing errors from N-GARCH (1,1) option pricing model

Moneyness S/K		Days-to-maturity					
		6-30	30 - 60	60-90	90-180	180-270	> 270
< 0.94	AP	1.25	4.92	10.09	22.22	45.07	89.82
	AE	1.05	2.86	6.02	17.56	27.71	45.06
	APE	93.73%	67.23%	75.97%	60.90%	51.66%	56.37%
	PE	56.93%	63.23%	75.97%	60.89%	51.66%	56.37%
0.94 - 0.97	AP	3.72	12.94	24.87	47.24	80.72	152.81
	AE	2.69	7.01	11.20	20.24	20.27	35.92
	APE	81.66%	55.75%	47.94%	36.68%	25.12%	21.44%
	PE	47.11%	54.35%	47.54%	36.21%	24.06%	20.44%
0.97-1.00	AP	14.03	33.48	50.23	76.27	111.71	201.98
	AE	5.80	6.04	8.77	13.16	25.61	53.85
	APE	76.66%	26.71%	22.20%	17.50%	17.54%	22.20%
	PE	46.43%	18.71%	15.20%	12.98%	16.10%	22.20%
1.00 - 1.03	AP	49.67	68.97	85.11	112.08	146.23	262.25
	AE	0.52	5.40	11.57	20.35	28.71	54.27
	APE	18.93%	13.90%	14.52%	15.10%	15.53%	19.10%
	PE	7.01%	8.07%	11.52%	13.98%	14.57%	19.06%
1.03 - 1.06	AP	104.87	116.31	129.43	156.65	189.79	312.19
	AE	-10.92	0.67	6.67	18.30	27.58	49.01
	APE	9.24%	9.10%	7.08%	11.33%	13.11%	16.51%
	PE	-5.16%	-5.62%	-1.82%	9.89%	12.60%	16.36%
> 1.06	AP	256.05	239.94	257.55	262.44	273.28	442.83
	AE	-19.04	-14.02	-6.30	3.51	16.44	35.69
	APE	5.65%	4.82%	4.25%	4.83%	6.47%	9.04%
	PE	-4.65%	-3.23%	-0.61%	2.51%	5.47%	8.80%

Summary and Conclusions

We have presented two alternative option pricing models that admit stochastic volatility in discrete-time framework - the so-called N-GARCH model with the property of nonlinear general autoregressive conditional heteroskedasticity and random jumps. It is shown that this approximation of closed-form pricing formulas is practically implementable, and it has made it relatively straightforward to study the relative empirical performance of two models of distinct interest. The parameters needed to imply those models in pricing options were computed using the maximium likelihood method.

The paper has shown that regardless of performance yardstick, the jump-diffusion model and N-GARCH model are still significantly misspecified. According to the out-of-sample pricing measures, ading the random-jump feature to the random walk model of asset return can further improve its performance, especially in pricing short-term options; whereas modeling stochastic volatility under a faily general GARCH process can enhance the fit of long-term options.

Notes

[1] See Bollerslev, Chou, Kroner (1992) for a survey of this literature.

[2] Mathematically speaking, the market price for additional risk is associated with the Girsanov transformation of the underlying probability measure leading to a particular martingale measure.

[3] See Merton (1973), p. 150.

[4] For more details about the Hamilton-Jacobi-Bellman equation we refer to Tomas Bjork (1998), Oksendal (1985).

[5] The log utility function correnspondens to the case of g = 1.

[6] Note that the relationships between the cumulants k_i with i=1, 2, 3, 4 of a distribution and its moments, are given by

$$\kappa_1 = E[Z(t)]$$

$$\kappa_2 = E\left[(Z(t))^2\right] - (E[Z(t)])^2$$

$$\kappa_3 = E\left[(Z(t))^3\right] - 3E[Z(t)]E\left[(Z(t))^2\right] + 2(E[Z(t)])^3$$

$$\kappa_4 = E\left[(Z(t))^4\right] - 3E\left[(Z(t))^2\right] - 4E[Z(t)]E\left[(Z(t))^2\right] + 12(E[Z(t)])^2 E\left[(Z(t))^2\right] - 6(E[Z(t)])^4$$

[7] The model simplifies to the popular GARCH model of Bollerslev (1986) when this correlation is absent.

[8] According to Duan (1995), the assumption $b_1 + b_2[1+(l + q)^2] < 1$ should be hold, then the stochastic process in (38) has the first order weak stationarity. The stationary expected value (or unconditional variance) of the process under consideration is equal to $\beta_0 \left\{1 - \beta_1 - \beta_2\left[1+(\theta+\lambda)^2\right]\right\}^{-1}$. The initial variance v_0 then could be set based on the unconditional variance of the process.

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