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A CHAPTER IN THE HISTORY OF THE RENAISSANCE MATHEMATICS: NEGATIVE NUMBERS AND THE FORMULATION OF THE LAW OF SIGNS (FERRARA, ITALY *CA.* 1450)*

As the process of the extension of the concept of number, from "natural number" of the Ancients toward "real number" of the twentieth-century mathematicians, has still not been completely explored, it is natural, therefore, that such topics as the use of irrational and of negative numbers, the latter ones being of our particular interest here, returns from time to time in publications on the history of mathematics. In this way the deeper insight into the development of the concept of numbers theory is given, thanks to discoveries of new sources or thanks to original interpretations of the sources published already. The studies revealing the presence and function of negative numbers in the "pre-Cartesian" mathematics are particularily interesting.

Among publications issued during the last decennia there are some that deal with the use of negative numbers before the seventeenth century. Jacques Sesiano's *The Appearance of Negative Solutions in Mediaeval Mathematics* (1985) points out the fact that the introduction of negative numbers into mathematics was the consequence of the admission of negative solutions (and not just of negative values) by mathematicians. What follows Sesiano's statement is a very thorough study of linear equations with negative solutions present in mathematical writings for more than one thousand years, from the *Arithmetica* of Diophantus (III century) to the *Summa* of Luca Pacioli (1494)[1].

Incidentally, the fact that the negative solutions were "present" in the arithematical practice it does not, of course mean that they were "accepted" in mathe-

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matics as a consequence of the "extended concept of number". Nevertheless, the gradual "insinuation" of negative numbers into mathematics through the problems of commercial arithmetic, the process that may be seen, for instance, in Leonardo Fibonacci's *Trattato d'abaco* (1202, 1228) or in Luca Pacioli's *Arithmetica* (1476-1480) and *Summa* (1494), led eventually to the "legal" acceptance of negative numbers, as it is explicit in Michael Stifels *Arithmetica integra* (1544).

Comming back to J. Sesiano's study: at the very beginning of it the author mentions the law of signs that, "not surprisingly, appeared at an early stage in the development of mathematics". Then Diophantus' *Arithmetica* is referred to, with its unique problem leading to a negative solution, and with the explanation of the law of signs in connection with added and subtracted quantities. Then there follow chapters on Indian, Chinese and Arabic mathematics, then the negative solutions in the Latin medieval tradition are considered, beginning with the treatise by Pseudo-Beda, before the Xth century. In this treatise negative numbers are meant "not as subtracted quantities", as it was the case in Diophantus' *Arithmetica*, but as "proper entities"[2].

As it results from Sesiano's study, further developments of mathematics, as present in Leonardo Fibonacci's works and in the works by the Provancale mathematicians of the first half of the fifteenth century, and then in the Nicolas Chuquet's and Luca Paccioli's achievements at the end of the said century, do not seem to overpass essentially the level attained in the Pseudo-Beda treatise. The same being true for the application of the law of signs, this law being applied to two operations: addition and subtraction (with the exception, however, of Luca Pacioli who used the rule of signs for and division as well)[3].

There exists another contribution to the subject, published ten years before the one considered above. It is Menso Folkerts' *Die Kenntnis der negativen Zahlen in Westeuropa bis zum 16. Jahrhundert*, in which the enunciation of the law of signs – this time plainly applicable also to multiplication and division, and expressed by symbols "+" and "–", is located in the first decennia of the sixteenth century (although, as Folkerts admits, the negative numbers have been "systematically used" since the end of the fifteenth century on). According to scriptural evidence referred to by Folkerts, namely the ms. Vienna 5277, f.2ss, written between 1500 and 1520, it was only then that the law of signs was explicitely stated[4].

The research, however, done recently on the fifteenth-century mathematical manuscripts, preserved, among others, in the Cracow University Library (Biblioteka Jagiellońska), reveals that the law of signs had been formulated, geometrically proved and currently used more than fifty years before the text in the ms. Wien 5277 was written, and some forty years before Pacioli's *Summa* was published. This law appears in a treatise on arithmetic by Giovanni Bianchini (*ca.*1400–*ca.*1470)[5].

Bianchini's mathematical background came probably from Venice where he spent his youth, from one of the *scuole d'abaco* that flourished there by the turn of the fourteenth and fifteenth centuries. In the after-Venetian period of his life (1427–1457), Bianchini was the administrator of the estate of the marquess d'Este and astronomer at their court, finding himself in the centre of intellectual and artistic life of the Renaissance Ferrara[6]. After retirement from the administration duties in 1457, he continued for some ten years his work on mathematical astronomy living either in Ferrara or on his rustic property.

1. GIOVANNI BIANCHINI'S *DE ARITHMETICA*

Bianchini's *De arithmetica* is incorporated into the astronomical work, the *Flores Almagesti*, composed in Ferrara during some twenty years, between the early forties and early sixties. About 1464 the *Flores* were known in Venice, consulted there by Regiomontanus[7]. The manuscript copies of the *Flores* preserved till now functioned in the second half of the fifteenth century in such university milieus as Ferrara, Bologna, Perugia and Cracow[8]. One of the copies was done in Naples[9]. The *Flores Almagesti*, that are devoted to mathematical astronomy, begin with treatises on arithmetic, algebra, proportions and the setting of the sine tables, considered as an introduction to astronomical part of the *opus* dealing with the *primum mobile*, planetary astronomy and eclipses. The *De arithmetica*, the subject of our present considerations, that subsequently will be referred to as *Arithmetic*, opens the *Flores Almagesti*. In fact, it was treated by Bianchini as the basis of the whole work. In this way the *Arithmetic* itself, that is frequently referred to in the other parts of the *opus*, functions itself independently from the rest of the *Flores*.

The *Arithmetic* is composed of an Introduction, twenty-one chapters, and three appendices[10]. In the Introduction the principles of the decimal positional system are explained, then in the first seven chapters the rules of multiplication, division, addition and subtraction of positive whole numbers and fractions are given. The remaining fourteen chapters, from eight to twenty one, deal with roots: extraction of quadratic and cubic roots, the *radices surdae* included, and four operations on quadratic roots of both kind (*in numeris* and *in quantitate continua*) with the application of the rule of signs[11]. The appendices are devoted to quadratic numbers, to proportions, and to operations on large numbers.

While the first part of Bianchini's *Arithmetic*, embracing chapters one through twelve, deals with operations on numbers considered as "concrete" or in other words positive whole numbers and fractions, the second part of it, *i.e.* chapters thirteen through twenty one, deals with numbers considered "abstractly", as abstract expressions of quantities. The latter approach towards numbers permits Bianchini to overpasse the well known troubles of psychological and philosophical nature, concerning negative and irrational numbers considered as a subject of mathematics. Bianchini's "continuous numbers" (*numeri continui* and *quantitates continuae*), in antiquity were commonly called *numeri irrationales* and in the Middle Ages *numeri surdi*, "deaf numbers" or numbers lacking a proper name (the term derived directly from the Muslim tradition), while the negative numbers were called *numeri absurdi* or even *numeri inexistentes*.

The subject of our further considerations will be the "part two" of Bianchini's treatise.

2. CHAPTER THIRTEEN OF THE *ARITHMETIC*: THE FORMULATION OF THE LAW OF SIGNS AND ITS CONTEXT. THE NATURE OF NEGATIVE NUMBERS

Chapter thirteen of Bianchini's *Arithmetic*, entitled *De practica radicum adinvicem* (Operations on roots), follows immediately the section of the treatise devoted to the extraction of the square and cubic roots in whole numbers and in *surds* (*in quantitate continua* or *in quantitate composita*). Chapter thirteen, principally dealing with multiplication and division of roots, is composed as follows: first comes the definition of multiplication, then the law of signs is given, which is followed by the rules of multiplication and division of roots by roots and of roots by numbers.

The definition of multiplication, opening chapter thirteen, signals through its elementary character that another part of the treaty begins right here. Since it is different from what preceded because it does not deal with positive integers and fractions any more, for this reason the basic rules have to be reintroduced.

The rules run as follows:

[1] "In multiplication three [elements] are required: multiplier, multiplicand and product. The proof of that is division: if the product is divided by the multiplier, the result is the multiplicand and vice versa. [...] And this – Bianchini concludes after giving also the geometrical proof – is valid for discret quantity"[12].

As for the rule of multiplication and division of "continuous quantities", Bianchini while proving it limits himself to a geometrical proof only. This limitation had been justified by him earlier, in chapter twelve: *De radicibus surdis in quantitate continua inveniendis,* where he explained the specific nature of quadratic incommensurabilities (continuous quantities) by means of the very ancient geometrical procedure, subsequently used also by Bombelli and by Descartes[13].

[2] "When a straight line is multiplied by another straight line equal to the first one, the product is a square of equal sides and right angles; a straight line multiplied by a straight line of a different length produces a quadrangle of right angles and the opposite sides parallel and equal to each other"[14].

The introductory explanations are followed by a paragraph that is the core of the chapter and of the whole treatise. Here the law of signs is formulated with respect to multiplication. In the remaining chapters of the *Arithmetic* this law will be enlarged to the addition and subtraction of negative numbers, as performed on "discrete" numbers as well as on "continuous" ones.

The law of signs:

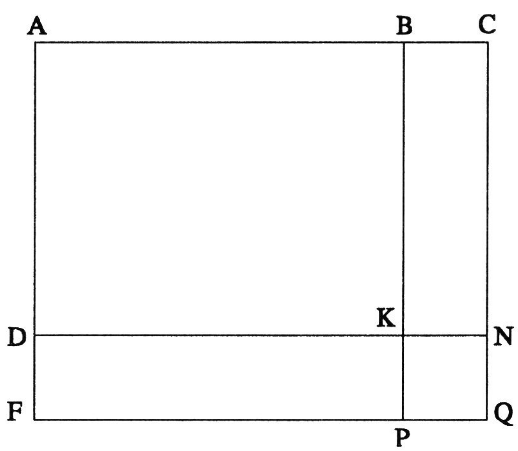
[1]. "When plus is multiplied by plus the product will be plus. This is obvious.

[2]. When plus is multiplied by minus, or minus by plus, the product will be minus and this results [from the fact] that more minus increases or plus decreases, the more "the product minus" [the negative product] increases.

[3]. When minus is multiplied by minus the product will be plus, because the more minus diminishes, the more plus increases"[15].

At this point Bianchini changes the subject and passes to operations on irrationales. This is only in the following chapters, devoted to operations on negative numbers, that he will supply the law of signs with geometrical proofs.

Now we will look closely at the statement [3]. Its "geometrical proof" is given in chapter eighteen, where Bianchini considers the multiplication: $(12-\sqrt{25})(10-\sqrt{9})$.



$AC = DN = FQ = 10$
$DFNQ + BCPQ = 86$
$BC = \sqrt{9}$
$DFKP + BCKN + KPNQ = 71$
$AF = 12$
$AB = 10 - \sqrt{9} = 7$
$DF = \sqrt{25}$
$AD = 12 - \sqrt{25} = 7$
$AFCQ = 120$
$AB \times AD = 49$

The proof is inspired by the Euclidean algebra, although the procedure proposed by Bianchini overpasses the *Elements* II[16].The essence of the proof, as it is pointed out by Bianchini, consists in the fact that the surface KNPQ, resulting from the multiplication of minus by minus, *i.e.* minus $\sqrt{25}$ by minus $\sqrt{9}$, is taken t w i c e in the course of the operation. In other terms, when the rectangles DFNQ and BCPQ are added to each other, the rectangle KNPQ is a d d e d r e p e a t e d l y. Incidentally, the Bianchini's interpretation of the multiplication minus by minus evokes the Maclaurin's (1748) "naive interpretation" of multiplication plus by minus, conceived as a repeated subtraction. The tentatives, aiming at the justification of the application of the laws, proper to operations on whole positive numbers, to operations on negative numbers and irrational ones, were undertaken also by the Maclaurin's contemporaries[17].

In the last part of the law [3] one more "incorrectness" or at least "a curiosity" is present, namely, the Bianchini's law of signs, the statement of the arithmetical order proved geometrically, finds its ultimate justification radicated in the intuition of the first principles. (The latter procedure, as the one mentioned above with reference to Maclaurin, also used by the eighteenth-century mathematicians)[18]. In the case of Bianchini, this is precisely the ultimate justification of the law of signs [3], now judged illegitimate from the point of view of the mathematical formalism, that shows the nature of Bianchini's concept of number. In fact, Bianchini's statement that "the more the negative number diminishes, the more the positive one increases", and thus "when minus is multiplied by minus the result is plus", is based on a number considered as *quantity* or even as an *abstract expression of quantity*, and not as *entity* any more.

Even if this is not the right place to consider the background of such a concept of number present in Bianchini's law of signs, some suggestions on the subject may be expressed taking account of Bianchini's professional activity. Knowing Bianchini's involvement in the commercial arithmetic on the one hand, and in mathematical astronomy on the other (he was a skillful calculator of mathematical and astronomical tables), one may be inclined to suppose that even if the concept of number used by Bianchini in the *Arithmetic* might seem to have resulted from Bianchini's philosophical background, Bianchini himself could perfectly do without philosophy. In the course of the calculation of the mathematical and astronomical tables he used to deal with irrational numbers, and while accomplishing his duties as a bookeeper, he used to deal with the notion of "debt" expressed by negative numbers. Bianchini, an *arithmeticus* by natural inclination, and a *logisticus* by the training received in a *scuola d'abbaco*, was interested in the notion of number large enough to denote equally positive and negative: whole numbers, fractions and irrationals.

In this context Bianchini's operations on irrationals, his consideration of subtracted quantities as negative numbers, as well as his law of signs presented in modern terms, even if lacking the use of symbols "+" and "–", evoke the approach towards numbers, particularly towards negative numbers, as present in the sixteenth and seventeenth century mathematics rather than the efforts in facing negative numbers undertaken by the fifteenth century mathematicians. Some twenty years after Bianchini's death Luca Pacioli asserted in his *Summa* the impossibility of subtracting $\sqrt{5}$ from $\sqrt{3}$, unless one talks of it abusively – "abusive parlando", "abusive loquando"; J. Sesiano characterizes Pacioli's attitude towards negative numbers as ambiguous "instinctively he refuses them, yet he considers the occurrence in an abstract problem of a negative solution to be a *bellissimo caso...*"[19].

Although the law of signs was commonly used from the half of the sixteenth century on, its part concerning the positive result of the multiplication of two negative numbers ("proved" and then accepted as obvious by Bianchini) stupefied mathematicians. In the eigheenth century some of them (Newton, Carnot), avoided proving the law of signs, conscious of inadequacy of the then assumed concept of number for such a logical operation. Consequently, there were mathematicians that operated on negative numbers, but did not accept them formally[20].

3. CHAPTERS SIXTEEN THROUGH NINETEEN: THE LAW OF SIGNS AS APPLIED TO MULTIPLICATION AND DIVISION

The chapters of the *Arithmetic* devoted to the use of the law of signs are composed in the following way: first the examples are given, then come geometrical proofs, finaly "the rule" is formulated. Actually, what Bianchini calls "the rule" is a step by step description of operations on all kind of numbers.

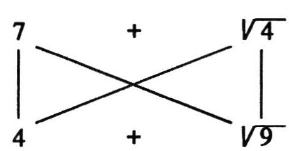
Subsequently Bianchini's examples will be given in the modern notation. Bianchini himself did not use symbols in his *Arithmetic.*

Chapter sixteen. Multiplication plus by plus.

The example: $(7 + \sqrt{4})(4 + \sqrt{9})$ is solved according to the formulas:

$$\sqrt{a} \,.\, \sqrt{b} = \sqrt{ab} \text{ and } a \,.\, \sqrt{b} = \sqrt{a^2 \,.\, b},$$

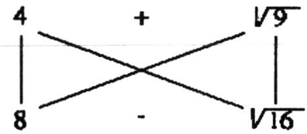
and the operations are performed following the order established by the "rule of the cross".



The rule: "I will begin [with] the left side, multiplying 7 by 4. The product will be 28. Then I will multiply crosswise the plus root of 4 by 4, what produces [the root of] 64. Then 7 by the plus root of 9 will produce the root of 441. Finally, on the right side I will multiply the plus root of 4 by the plus root of 9, what produces the plus root of 36. All these roots joined together will be [equal to] 35 in numbers, which added to 28 of the first multiplication will give the sum equal to 63"[21].

Chapter seventeen. Multiplication plus by minus:

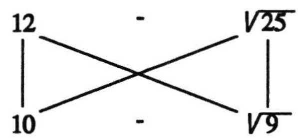
Example: $(4+\sqrt{9})(8-\sqrt{16})$



The rule: "Multiply 4 by 8. The product will be 32. Then 8 by the plus root of 9 produces the plus root of 576. Then 4 by the [minus] root of 16 produces the minus root of 256. Then the plus root of 9 by the minus root of 16 produces the minus root of 144. So add 32 to the root of 576, that is 24, the sum will be 56. From that you subtract the roots of 256 and of 144 – they are [equal to] 28. What remains is also 28[22].

Chapter eighteen. Multiplication minus by minus:

Example: $(12-\sqrt{25})(10-\sqrt{9})$



The rule: "Multiply 10 by 12 the product will be 120. Multiply also 12 by the minus root of 9 and this will produce the minus root of 1396. Then 10 by the minus root of 25 produces the minus root of 2500. Then the minus root of 25 by the minus root of 9 produces the plus root of 225. So add the root of 225, that is 15, to 120, the sum will be 135. From this subtract the root of 1296, that is 36, and the root of 2500, that is 50. The sum that has to be subtracted is 136. Remains 49"[23].

In Chapter nineteen Bianchini proposes solutions of the following examples: $50 : (\sqrt{36}+\sqrt{16})$ and $\sqrt{10000} : (\sqrt{100}-\sqrt{25})$, but as the formulas transmitted in manuscripts are wrong, the result obtained in the second case is $\sqrt{177}\ 7/9 + \sqrt{44}\ 4/9$ [*sic*].

4. CHAPTERS FOURTEEN, FIFTEEN AND TWENTY: THE RULE OF SIGNS AS APPLIED TO ADDITION AND SUBTRACTION OF IRRATIONALS. NEGATIVE (INTERMEDIATE) RESULT

In these chapters, while solving examples of addition and subtraction of irrationals, Bianchini does not evaluate the results, but considers the *radicandae* analogically to the algebraical symbols.

Chapter fourteen: Addition of roots.

In this chapter the addition of irrationals is done according to the formula: $\sqrt{a} + \sqrt{b} = \sqrt{a + b + \sqrt{4ab}}$,, and the following examples are given: $\sqrt{24} + \sqrt{6}$ and $\sqrt{5} + \sqrt{8}$.

First example:

The rule: "When you add the root of 24 to the root of 6 [first] multiply 24 by 6, the product will be 144. And [then] double the root of it and the [result is] the root of 576, which is 24. Afterwards add 24 to 6, which is 30, and then add 24. The sum is 54. The root of 54 is the [result] of the aggregation of the root of 24 to the root of 6"[24].

Second example:

"If you want to add the root of 5 to the root of 8, multiply 5 by 8, the product will be the root of 40, which when doubled – according to the former [rule] – gives the result equal to the root of 160. In the same way add 5 to 8, which will be 13. Thus it is not possible to answer otherwise than: the result is the root of [the root] of 160 plus 13 [$\sqrt{\sqrt{160} + 13}$]"[25].

Chapter fifteen: Subtraction of irrationals.

The subtraction of irrationals is done according to the formula $\sqrt{a} - \sqrt{b} = \sqrt{a + b - \sqrt{4ab}}$, and the following examples are given: $\sqrt{54} - \sqrt{6}$ and $\sqrt{4} - \sqrt{8}$.

The rule: "I multiply 6 by 54, the product is 324. The root of it is 18, what I double and it is 36. Add 54 to 6, the sum will be 60. Subtract 36 from 60, remains 24. The solution is the root of 24"[26].

Second example:

"If you want to subtract the root of 4 from the root of 8, multiply 4 by 8, the product will be 32. The double of it is the root of 128. Add also 4 to 8. The sum will be 12. The answer must be that the root of what results from the subtraction of the root of 128 from 12 is the [solution], and there is no other way to answer"[27].

Chapter twenty: Additions and subtractions of the *compositae.*

The term *compositae* – composite irrationalities – is used by Bianchini in two senses. Firstly it means rationals and irrationals while both of them involved in aritmetical operations, secondly numbers with irrational components[28].

Bianchini's examples are as follows:

$$(15 + \sqrt{8}) + (\sqrt{50} - 5) \text{ and } (50 + \sqrt{6}) - (55 - \sqrt{24})$$

The first example provided Bianchini with the opportunity to demonstrate the order in which the operations should be performed. It is always the order resulting from the "rule of the cross". In this way first comes the addition of roots to roots, then the addition of numbers to numbers. N.B. here also irrationals function analogically to unknows in algebra.

In the second example, directly pertinent to our considerations, the values seem to be carefully chosen by Bianchini: its solution not only requires the subtraction of a negative number but also the subtraction of a greater number from a smaller one, what leads to a negative solution. Bianchini applies here the rule inspired by algebra and states as follows:

"Note that when you happen to subtract a number from a number or a root from a root, to which other numbers or roots are added or subtracted, you always have to add [the quantity] that was subtracted from one side to the other side, and only after that you proceed with subtraction as it is [shown] in the example: [*i.e.* $(50+\sqrt{6})-(55-\sqrt{24})$, the remark concerns $\sqrt{6}-\sqrt{24}$]. In the subtrahend there is the minus root of 24. Thus [the root of 24] is the [value] that has to be added to the root of 6. The sum, according to the rule from chapter 14, is the root of 54. Then you have to subtract 50 plus the root of 54 from 50; but as the number that has to be subtracted exceeds the number from which it has to be subtracted, thus you have to proceed in the opposite way: subtract 50 from 55, remains 5, and it has to be answered that there remains the root of 54 minus 5. And this is the result"[29].

5. CHAPTER TWENTY ONE: OPERATIONS ON MIXED SURDS. THE SUBSTRACTING ADDITION

In this chapter, concluding the treatise, Bianchini provides the reader with some complementary instructions, according with the title "The other doctrines necessary in [dealing with] composed numbers". Bianchini introduces the subject as follows:

"I want also to teach you what happens when together with numbers and roots there are other numbers or roots, added or subtracted; [how] one has to work while adding or subtracting successively, or else multiplying or dividing; how [you ought to] help yourself – because I have never seen others explain it. According to this teaching and thanks to the subtleness of your mind, you will be able to give the proper and right explanation to the questions which others, ignorants, are not able to answer"[30].

The instructions deal with the application of the associative law of addition, applied equally to the *additio minuenda*, the distributive law, and the associative law of multiplication.

The following examples are considered:

$$(20-\sqrt{36})+3 \qquad (20-\sqrt{36})-(-3)$$

$$(\sqrt{15}+18+\sqrt{135})(\sqrt{18}-\sqrt{36}-\sqrt{8}) \qquad (\sqrt{7}-8-\sqrt{63})(20-\sqrt{28}+6)$$

$$(\sqrt{15}+18-\sqrt{135})(18-\sqrt{36}-\sqrt{8}) \qquad (\sqrt{7}-8-\sqrt{63})+(20-\sqrt{28}+6)$$

While solving the first two examples: $(20-\sqrt{36})+3$ and $(20-\sqrt{36})-(-3)$ Bianchini expresses the law of signs for addition and subtraction of negative numbers.

"[...] when it is said: 20 minus the roots of 36 subtracted [-3] or added 3, [in such cases] the less the subtrahend increases in subtraction, the more the rest increases. Thus: when the *subtracting addition* is applied to a negative number the entire number diminishes, as I have explained before, in the definition and demonstration of the multiplication minus by minus and plus by plus”[31].

[Thus in the second case], ”you have to add 3 to 20, [the result] will be 23. So 20 minus the root of 36 subtracted minus 3 is equal to 23 minus the root of 36 or [else] subtract 3 from 20, the result is [always] 17; so the same is 20 minus the root of 36 added 3 [*i.e.* $20 - (\sqrt{36} + 3)$] and 17 minus the root of 36. And as this is obvious, I don’t want to extend myself into similar considerations”[32].

Remarks concerning the solution of the third example:

”According to the previous teaching we have to add the root of 15 to the root of 135, and by [the rule in chapter] 14 of this treatise the sum will be the root of 240. Thus the first multiplicand is the root of 240 plus 18.

In the similar way you will add the root of 8 to the root of 18 and get the root of 50. Thus the [second] multiplicand is the root of 50 minus the root of 36. Multiply the root of 240 plus 18 by the root of 50 minus the root of 36. You will have the solution following the rules in [chapter] fifteen of this [treatise][33].

CONCLUSIONS

In the *Arithmetic* Bianchini passes from concrete problems, as it was the case in his *Algebra*, to abstract ones, and while solving them, he uses numbers considered as an (abstract) expression of quantity. Moreover, he formulates the law of signs explicitly and in modern terms, and applies it to the four arithmetical operations. The subtraction of a bigger number from a smaller one is for Bianchini a purely formal question, a negative result of such an operation being acceptet by him as equally justified as it would be, reversing the procedure, a positive one. In the course of some of Bianchini’s calculations the subtracted quantities function as negative numbers.

Bianchini’s proof of the law of signs (formally incorrect) was inspired by the same principles by which were inspired the proofs of this law elaborated by some of the eighteenth- century mathematicians.

Notes

* The Harvard University Center for Italian Renaissance Studies, Villa I Tatti, Florence and the Centro Studi e Incontri Europei, Roma, via Anicia 12, enabled me to study Bianchini’s manuscripts preserved in Italy. The study of Bianchini’s manuscripts in the Bibliotheque Nationale, Paris, was possible thanks to stipends accorded by the CNRS and facilities procured by the Observatoire de Paris and its ”Equipe pour l’histoire de l’astronomie”. In the final stage of the work the Library of the Sidney M. Edelstein Center for

the History and Philosophy of Science, Medecine and Technology, Jerusalem, was very useful.

[1] Jacques S e s i a n o: *The Appearance of Negative Solutions in Madiaeval Mathematics.* "Archive for the History of Exact Sciences" T. 32, 1984 p.107.

[2] Menso Fo l k e r t s: *Pseudo-Beda. De arithmeticis propositionibus. Eine mathematische Schrift aus der Karolingerzeit.* "Sudhoffs Archiv" T. 56, 1982 pp.22–43. The negative solutions in the Muslim mediaeval mathematics are considered in Roshdi R a s h e d: *L'Arithmétisation de l'algèbre au 11ème siècle.* In: *Proceedings of XIIIth International Congress of the History of Science – Moscow, August 18–24 1971* Sections III, IV, Moscow 1974 p.66–67.

[3] Pacioli did not divide, however, minus by minus. See J. S e s i a n o, op.cit., pp. 142–147.

[4] Menso F o l k e r t s: *Die Kenntnis der negative Zahlen in Westeuropa bis zum 16. Jahrhundert.* In: *Proceedings of XIIIth International Congress of the History of Science* op. cit., particularly p.89 and p. 93, note 25, in which M. Folkerts, following W. Kaunzner's discovery, refers himself to the ms. Vienna 5277.

[5] G. Federici V e s c o v i n i: *Bianchini Giovanni.* In: *Dizionario biografico degli Italiani,* vol.X, Roma 1968, pp.194–196. The fourther contributions to Bianchini's scientific biography see A. G e r l, *Trigonometrisch-astronomisches Rechnen kurtz vor Copernicus. Der Briefvechsel Regiomontanus-Bianchini,* (Boethius, vol.XXI), Stuttgart 1989, pp.20–26. G. R o s i ń s k a: *Tables of Decimal Trigonometric Functions from ca. 1450* to *ca. 1550.* In: *From Deferent to Equant. A Volume of Studies in the History of Science in the Ancient and Medieval Near East in Honor of E.S. Kennedy.* Ed. by D.A. K i n g and G. S a l i b a. Annals of the New York Academy of Sciences. Vol. 500, 1987 pp. 419–426. E a d e m: *Giovanni Bianchini - matematyk i astronom XV wieku. (Giovanni Bianchini - Mathematician and Astronomer of the Fifteenth Century.* In: „Kwartalnik Historii Nauki i Techniki". T. 26, p.555 (with an English summary). E a d e m, *Algebra w środowisku astronomów krakowskich w XV wieku. Traktat z Flores Almagesti Jana Bianchiniego. (The Algebra of Giovanni Bianchini in the Milieu of the Cracow Astronomers. (XVth Century).* „Kwartalnik Historii Nauki i Techniki". T. 39, 1994 No 2, pp.3–19 (English summary). See also E. P o u l l e, *La bibliothèque scientifique...* quoted below, note nr.9.

[6] *La rinascita del sapere. Libri e maestri dello studio ferrarese.* Venezia 1991.

[7] Maximilian C u r t z e: *Der Briefwechsel Regiomontans mit Giovanni Bianchini, Jacob von Speyer und Christian Roder.* In: *Abhandlungen zur Geschichte der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen.* Vol. 12, 1902 p.206.

[8] The ms Vaticanus Lat. 2228 was copied in Ferrara, in 1470. With Bologna was linked the ms preserved in the Biblioteca Universitaria, Bologna, BU 19(293), and with Perugia the ms now in Biblioteca Palatina Augusta, Perugia ms 1004. These codices, together with the manuscripts containing Bianchini's astronomical tables and the *Canones tabularum* are subject of publications by Lynn T h o r n d i k e: *Giovanni Bianchini in Paris Manuscripts.* "Scripta Mathematica" T. 16, 1950 pp.5–12 and 176-180 and I d e m: *Giovanni Bianchini in Italian Manuscripts.* "Scripta Mathematica" T. 19, 1953 pp.5–17. The ms BJ 558, preserved in the Jagiellonian Library (Biblioteka Jagiellońska), Cracow,

annotated by Regiomontanus, was probably the propriety of Martinus Bylica of Olkusz, professor at Cracow University and colleague of Regiomontanus. See G. R o s i ń s k a, *Algebra w środowisku...* op.cit., p.17 notes 37–40.

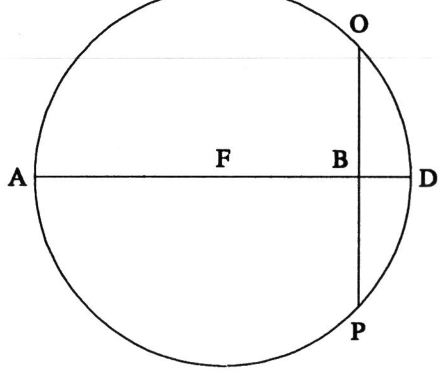
[9] This copy, now in the Bibliotheque Nationale, Paris, BN lat. 10253, was done by Arnoldus de Bruxella in the eightieths of the fifteenth century. See Emmanuel P o u l l e: *La bibliothèque scientifique d'un imprimeur humaniste au XVe siècle. Catalogue des manuscrits d'Arnauld de Bruxelles à la Bibliothèque Nationale de Paris.* In: *Travaux d'Humanisme et Renaissance.* T.LVII Génève (Droz) 1963 pp.38–44.

[10] Prohemium. Cap.1. De regula multiplicandi. Cap.2. De practica dividendi. Cap.3. De practica multiplicandi in fractionibus. Cap.4. De practica dividendi in fractionibus. Cap.5. De aggregatione fractionum. Cap.6. De subtractione fractionum. Cap.7. Regula infilzandi secundum vulgares qua operatur in divisionibus fractionum. Cap.8. De practica in radicibus universalibus operanda. Cap.9. De tribus generibus radicum. Cap.10. De radicibus quadratis in numeris extrahendis. Cap.11. De radice cubica in numeris extrahenda. Cap.12. De radicibus surdis in quantitate continua inveniendis. Cap.13. Sequitur de practica radicum adinvicem. Cap.14. De aggregatione radicum. Cap.15. De subtractione radicum. Cap.16. De multiplicatione plus per plus. Cap.17. De multiplicatione plus per minus. Cap. 18. De multiplicatione minus per minus. Cap.19. De practica dividendi in compositis. Cap.20. De practica addendi atque subtrahendi in compositis. Cap.21. De caeteris doctrinis in compositis necessariis. [Appendices:] De numeris quadratis, De proportionibus, Regula facilis ad dividendum quemlibet numerum maximum per alium magnum. I refer myself to the critical edition of Bianchini's *Arithmetic*, forthcoming in the "Studia Copernicana" series.

[11] Bianchini mentions also the *radix relata* and *radix relatae relata.*

[12] In omni multiplicatione tria requiruntur: numerus multiplicans, secundo multiplicandus, et productum; cuius probatio est divisio, quia si productum dividatur per multiplicantem exibit multiplicandus et econverso, si per multiplicandum exibit multiplicans. Et hoc in quantitate discreta.

[13] De radicibus surdis in quantitate continua inveniendis [...]. Quaero ergo radicem de 24 in linea productam. Est ergo linea longitudinis 24 producta secundum mensuram tuam, sit ergo linea AB. Et continuabo ipsam a puncto B per quantitatem unius numeri secundum mensuram primo mensuratam, quae sit BD. Deinde totam lineam AD dividam in duabus partibus aequalibus in puncto F, super quem firmabo pedem circini et componam circulum secundum quantitatem diametri AFBD. Postmodum supra punctum B erigam perpendicularem usque ad circumferentiam, quae sit BO, quam etiam continuabo usque in P, et erit linea OBP. Dico enim quod linea BO est latus quadrati superficiei de 24, et hoc probatur per 24 secundi atque per 3 et 34 tertii [Euclidis]. Quia manifeste demonstratur per 3 tertii linea AD, quae transit per centrum circuli, dividit lineam OP in duabus aequalibus partibus in puncto B, et pet 34 eiusdem, quod sit ex DB in BA aequum est quadrato OB, quod est propositum.



$AB = 24$
$BD = 1$
$BO^2 = AB \times BD \quad BO = \sqrt{24}$

Cf. Wilbur R. K n o r r: *The Evolution of the Euclidean Elements. A Study of the Theory of Incommensurable Magnitudes and Its Significance for Early Greek Geometry*, (Reidel), Dordrecht/Boston 1975, pp.182 and 206, note 28. A r y a b h a t a t h e E l d e r: *The Aryabhatiya*, English translation in: *The Treasury of Mathematics*, Ed. by H.O. Midonick, New York (Philosophical Library), 1965, p.42. René D e s c a r t e s, *La Géometrie*, 1637, *Livre premier. Des problèmes qu'on peut construire sans y employer que des cercles et des lignes droites.*

[14] [...] quando linea recta multiplicatur per lineam rectam sibi aequalem producitur quadratum aequalium laterum et rectorum angulorum. Et linea recta multiplicata per lineam rectam sibi inaequalem producitur quadrangulum rectorum angulorum et laterum aequedistantium, cuius quelibet duo latera opposita sunt aequalia.

[15] Quando plus multiplicatur per plus productus erit plus et hoc clarum est. Quando plus multiplicatur per minus aut minus per plus productum erit minus et hoc patet, quia quanto minus augetur aut plus minuetur tanto productum fiet minus. Quando minus multiplicatur per minus productus erit plus, quia quantum minus minuitur tantum plus augetur.

[16] De multiplicatione minus per minus. Capitulum 18. Item volo multiplicare minus radix de 25 per 10 minus radix de 9. Producam lineam AF quae sit 12 et de ipsa resecabo FD quae sit radix de 25, et a puncto A ducam lineam ad angulum rectum quae sit ABC, cuius AC sit 10 et BC radix de 9. Quaero enim superficiem multiplicationis lineae AD per AB, quae est ADBK.

Perficiam superficiem totius multiplicationis AF per AC, cuius quadrangulum aequalium oppositorum laterum et angulorum rectorum erit AFCQ, cuius quantitas est 120. Et continuabo lineam DK usque ad punctum N, item BK usque ad P, et factae sunt tres superficies quadrilaterae rectorum angulorum et laterum contra se positorum aequalium et notorum, quarum quaerimus quantitates.

Quare DN quae est aequalis AC, id est 10, multiplicata per DF, quae est radix de 25, cuius superficies est radix de 2500, id est 50, et CQ quae est aequalis AF, quae est 12, quae multiplicata per radicem de 9 quantitatis lineae BC, cuius productum est radix de 1296, id est 36.

Inventae ergo sunt faciliter duae quantitates, quadranguli scilicet DFNQ, quae est 50, et quadranguli BCPQ, quae est 36, quarum summa est 86.

Sed ut intuentibus manifeste patet quadrangulum KPNQ duabus vicibus reperitur in quantitatibus supradictis, et hoc est propter quod multiplicando minus per minus productum erit plus. Oportet ergo quaerere ipsius quantitatem.

Manifestum est KP latus esse aequale DF id est radix de 25, et KN aequale BC quae est radix de 9. Quare superficies KPNQ erit radix de 225, quae est 15, qui subtrahendi sunt de 86, restant 71. Quare tres superficies quadrangulorum, scilicet DFKP, BCKN, KPNQ simul iunctae sunt in quantitate 71.

Et quia, ut supra scriptum est, tota superficies AFCQ est 120, ab ipsa quantitate dempta 71, restat superficies quadrilateri seu producti AD per DK 49, quod est propositum.

[17] A.P. J u s z k i e w i c z [J u s c h k e w i t s c h] Ed.: *Historia matematyki od czasów najdawniejszych do początku XIX stulecia*, Vol.II, *Matematyka XVII stulecia.* [Polish translation by S. Dobrzycki]. Warszawa 1976 pp.58–61.

[18] A.P. J u s z k i e w i c z: op.cit. p.59, where the L. Euler's proof of the rule of signs is considered.

[19] J. S e s i a n o: op.cit. p.143 note 75, and p.148.

[20] A.P. J u s z k i e w i c z: op.cit. pp.58, 61. I consider the subject more in detail in *Bianchini's theory of irrational numbers* (forthcoming).

[21] Volo multiplicare 7 et plus radicem de 4 per 4 et plus radicem de 9. Primo componam lineam multiplicantem ABC, cuius quantitas sit AB, videlicet 7, et BC radix de 4. Et a puncto A ad angulum rectum per undecimam primi [Euclidis] ducam lineam multiplicandam ADQ, cuius quantitas AD sit 4, et DQ radix sit de 9. Ex quibus duabus lineis perficiam quadrangulum ad angulos rectos per eandem, qui sit AQCP, cuius superficies, per id quod declaratum est per Euclidem in propositione prima libri secundi, est quod quaerimus, scilicet productus cuius demonstrabo quantitatem.

Primo a puncto B producam aequedistantem AQ per 31 primi libri [Euclidis] quae erit BT. Item a puncto D producam aequedistantem AP per eundem, quae est DO, quae duo aequedistantes sese intersecabunt in puncto M. Et est manifestum quod de superficie tota producti multiplicationis datae factae sunt 4 superficies quadrilaterae rectorum angulorum, quorum quelibet latera sunt nota, quia quodlibet ipsorum, contra se positi, sunt aequales et per consequens quorumlibet superficies sunt notae: AD latus est notum 4 et AB 7, quare superficies eius est 28. Item DQ est radix de 9 et QT est 7, quare superficies DE est radix de 441, quae est per numerum 21. Latus BC notum est radix de 4 et CO est 4, quare superficies BO est radix de 64, id est per numerum 8. MT etiam est radix de 9 et MO est radix de 4, quare superficies OT est radix de 36, id est 6. Quare quatuor superficiei quantitates simul iunctae constituunt in numero 63.

Patet ergo productum lineae AQ per lineam AC esse 63, quod est propositum. [...] Incipiam ad sinistram multiplicare 7 per 4 et fiet productum 28. Deinde per modum crucis multiplicando plus radix de 4 per 4 et producitur radix de 64. Item 7 per radicem de 9 plus et producitur radix de 441. Ultimo ad dexteram multiplicabo radicem de 4 plus per radicem de 9 plus et producitur radix de 36 plus, quae radices simul iunctae erunt in numeris 35, qui aggregati cum 28 primae multiplicationis fient in summa 63, quod est idem propositum.

[22] [...] Multiplica 4 per 8 erit productum 32. Item 8 per radicem de 9 plus producitur radix de 576 plus. Item 4 per radicem de 16 minus producitur radix de 256 minus. Item plus radix de 9 per minus radix de 16 producitur radix de 144 minus. Adde ergo 32 cum radice de 576, quae est 24, erit eorum summa 56, a quibus subtrahe radices de 256 et de 144, quae sunt 28, restant etiam 28, quod est idem propositum.

[23] [...] Multiplica 10 per 12 erit productum 120. Item multiplica 12 per radicem de 9 minus, producitur minus radix de 1296. Item 10 per radicem de 25 minus producitur radix de 2500 minus. Item radix de 25 minus per radix de 9 minus producitur radix de 225 plus. Adde ergo radix de 225, quae est 15, cum 120 fit aggregatum 125, a quibus deme radicem de 1296, quae est 36 et radicem de 2500, quae est 50. Quae summa demenda est 86. Restat 49, quod est idem propositum.

[24] Quando volueris addere radix de 24 cum radice de 6, multiplica 24 per 6 et producitur 144, cuius radix dupla et fiet eius duplum radix de 576 quae est 24. Postmodum adde 24 cum 6 sunt 30, quibus adde 24, eorum summa est 54, cuius radix est aggregatio radicis de 24 cum radice de 6.

[25] Et si volueris addere radicem de 5 cum radice de 8 multiplica 5 per 8 et producitur radix de 40, quae duplata per praecedentem provenit radix de 160. Item aggrega 5 cum 8 erunt 13, quia non potest aliter responderi nisi quod provenit radix aggregationis radicis de 160 cum 13.

[26] Multiplico 6 per 54 producitur 324, cuius radix est 18, quam duplabo et fiet 36. Item addo 6 cum 54 et fiet 60. Dempto ergo 36 de 60 restant 24, cuius radix est propositum.

[27] Et si volueris subtrahere radicem de 4 de radice de 8 multiplica 4 per 8 fiet productum 32, cuius duplum est radix de 128. Item aggrega 4 cum 8 erit summa 12. Quare respondendum est quod provenit radix residui subtrahendi radicis de 128 de 12. Nec aliter responderi potest.

[28] Galina Matvievskaya discusses the composite irrationalities in the commentary on Book X of the *Elements* by Ahmad ibn al-Husayn al-Ahwazi al-Katib (fl. about 940). Al-Ahwazi, apart from the "Bianchini's" composite irrationalities expressed as $a + \sqrt{b}$, where a is rational and b is not a square number, considers $\sqrt{a} + \sqrt{b}$, where neither a nor b are square numbers. G. Matvievskaya: *The theory of Quadratic Irrationals in Medieval Oriental Mathematics*. In: *From Deferent to Equant...* op.cit. p.263.

[29] Nota quod quando contingerit subtrahere numerum de numero aut radices de radicibus, cum quibus sunt alii numeri seu radices applicatae seu diminutae, semper debes illud quod diminutum est ab una parte, applicare alterae parti, et hoc facto sequere subtractionem.

Prout in proposito, in parte subtrahenda est radix de 24 diminuti quam debes applicare radici de 6. Cuius aggregatum, per 14 huius, est radix de 54. Debes ergo de 50 plus radice de 54 subtrahere 55. Et quia numerus subtrahendus excedit numerum a quo debes subtrahi, fac econverso, subtrahe 50 de 55, restant 5. Quare responderi debet quod restat radix de 54 minus 5. Et hoc est propositum.

[30] Volo te etiam instruere quod accidit quando cum numeris seu radicibus sunt alii numeri seu radices applicandi seu *demendi*. Et successive per ipsos oportet laborare in addendo seu minuendo aut multiplicando sive dividendo, quomodo te debes iuvare, quia

per alios numquam vidi hoc propalari, de hoc quod ex hac doctrina, cum subtilitate tui ingenii, ad multas quaestiones, quae per alios ignorantes non recte respondentur, tu iuste et recte definire potes.

[31] Et primo dico: quando dicitur 20 minus radice de 36 diminutis 3 vel auctis 3, quod quanto minus in diminutione augetur tanto numerus augetur, vel quando cum numero minuendo applicatur *additio minuenda*, totus numerus minuetur, ut in praecedentibus, in definitione et demonstratione multiplicationis minus per minus et plus per plus dixi.

[32] Quare addendi sunt 3 cum 20 et fient 23. Ergo idem est 20 minus radix de 36 additis 3 quantum 17 minus radice de 36 diminutis 3, quantum 23 minus radix de 36, vel minus 3 de 20, cuius residuum est 17. Quare idem est 20 minus radix de 36 additis 3, quantum 17 minus dadix de 36. Et quia hoc clarissimum est, nolo longius in similibus me in sermonem extendere.

[33] Volo multiplicare radix de 15 plus 18 diminuta radice de 135 per radicem de 18 minus radix de 136 diminuta radice de 8. Et per praecedentem doctrinam debemus aggregare radicem de 15 cum radice de 135 et per 14 huius fiet aggregatum radix de 240. Quare primus numerus multiplicandus est radix de 240 plus 18. Similique modo aggregabis radicem de 8 cum radice de 18 et fiet radix de 50 et factus est numerus multiplicans: radix de 50 minus radice de 36.

Multiplica ergo radicem de 240 plus 8 per radicem de 50 minus radix de 36. Per 15 huius habebis intentum.

Grażyna Rosińska

ROZDZIAŁ W DZIEJACH MATEMATYKI EPOKI RENESANSU: LICZBY UJEMNE ORAZ SFORMUŁOWANIE PRAWA ZNAKÓW (FERRARA, WŁOCHY OK.1450)

W historii matematyki zagadnienie bliższego wskazania daty wprowadzienia działań arytmetycznych na liczbach ujemnych jest dotąd sprawą otwartą. Na podstawie badań przeprowadzonych w ciągu ostatniego dwudziestolecia przyjmowano, że w kulturze zachodniej prawo znaków nie było stosowane wcześniej niż na przełomie XV i XVI wieku, a wyraźnie sformułowane, chociaż bez dowodów, dopiero w pierwszej ćwierci wieku XVI. Sporadycznie liczby ujemne pojawiały się w praktyce rachunkowej już w czasach karolińskich, a następnie u Leonarda Fibonacciego (początek XIII wieku), w postaci ujemnych wyników równań liniowych (por. przypisy 1, 2, 4). Analiza *Arytmetyki* Giovanniego Bianchiniego, napisanej między rokiem 1440 a 1450 (i zachowanej dotąd w sześciu egzemplarzach rękopiśmiennych, w tym jeden egzemplarz w Krakowie, por. przypisy 8, 9), pozwala na przesunięcie wstecz daty sformułowania prawa znaków o przeszło pół wieku. Przy tym prawo to podane jest przez Bianchiniego w brzmieniu takim, jak znamy je dzisiaj, bez użycia jednak symboli „+" i „–". Życiorys Bianchiniego wskazuje na włoskie środowisko "dworsko-handlowe", jako sprzyjające twórczej myśli matematycznej

Renesansu, potwierdzając znaczenie logistyki (technik obliczeniowych i reguł z nimi związanych) dla rozwoju arytmetyki.

O tym, że Bianchini był jednak nie tylko logistykiem, zresztą fenomenalnie sprawnym w dokonywaniu obliczeń, ale także twórczym arytmetykiem, świadczą skonstruowane przez niego "dowody" geometryczne prawa znaków, oparte na Euklidesie, ale wychodzące poza *Elementy*. Są to pierwsze dowody tego prawa w nauce europejskiej, a intuicje Bianchiniego, ujawnione przy ich okazji, były bliskie następnie matematykom epoki Oświecenia (przypisy 18, 20). Mówienie o liczbach ujemnych przy okazji omawiania działań dokonywanych na liczbach niewymiernych (niewymierności kwadratowe) ukazuje zasługi Bianchiniego dla rozszerzenia pojęcia liczby nie tylko o liczby ujemne, ale także o „wielkości niewspółmierne" traktowane tu jako liczby. Przy okazji wykładu o wielkościach niewspółmiernych (quantitates continuae, numeri continui) Bianchini podaje konstrukcję geometryczną, sygnalizowaną w europejskiej matematyce starożytnej, znaną matematyce Indii od V wieku po Chrystusie, lecz, jak sądzono dotychczas, w nowożytnej myśli europejskiej występującą dopiero w dziełach Bombellego, a następnie Kartezjusza (przyp. 13).