Rosińska, Grażyna

The "Fifteenth-Century Roots" of modern mathematics = (Matematyka XV wieku a początki matematyki nowożytnej)

1996

Grażyna Rosińska
(Warsaw)

THE "FIFTEENTH-CENTURY ROOTS" OF MODERN MATHEMATICS
THE UNIT SEGMENT. ITS FUNCTION IN BIANCHINI'S *DE ARITHMETICA*,
BOMBELLI'S *L'ALGEBRA*... AND DESCARTES' *LA GÉOMETRIE*

> Die große Aufgabe, welche seit der Pythagoreischen Entdeckung des Irrationalen
> gestellt ist, das uns (namentlich in der fließenden Zeit und der Bewegung) unmit-
> telbar anschaulich gegebene *Stetige* nach seinem in „exakten" Erkenntnissen for-
> mulierbaren Gehalt als Gesamtheit diskreter »Stadien« mathematisch zu erfassen, die-
> ses Problem ist trotz Dedekind, Cantor und Weierstrass heute so ungelöst wie je.
>
> H. Weyl, *Das Kontinuum*, 1918, p. 16.

INTRODUCTION

Since the roots of modern mathematics seem to be as old as mathematics itself – and by "roots" I mean the fundamental problems of mathematics raised in Antiquity, including the problem of continuity and thus also that of the magnitude-number relation – I decided to use quotation marks for the part of the title which would appear to locate the "roots of mathematics" in a period relatively close to our times. And since the answers to the "fundamental problems of mathematics" given by mathematicians over thousands of years have not appeared to be fully satisfactory to this day, I opened my discussion of the achievements of the 15th-century mathematician by a passage taken from Hermann Weyl's "Das Kontinuum", a work which is devoted to the critical situation of the foundations of mathematics, as it originated in Antiquity and as it persists even in the 20th century[1].

Nonetheless, according to a common assumption "modern mathematics" begins in the seventeenth century with Descartes' *La Géometrie* (1637), a work that approached the "task which has been facing us since the Pythagorean discovery of the irrationals" in a new way, and led eventually, together with Fermat's Isagoge, to the invention of the calculus. Particularly books Two and Three of *The Geometry* are regarded as the essence of Descartes' contribution to mathematics. In fact, the correspondence between the equation $f(x,y)=0$ and the locus satisfying this equation expressed by points having coordinates (x,y) relative to axes, is given there (p.ex. *The Geometry*, the second chapter of Book Two: "The method of distinguishing all curved lines of certain classes, and of knowing the ratios connecting their points on certain straight lines")[2]. So, as far as the idea of coordinates is considered to be the main point in the institution of analytic geometry, the appreciation of Books Two and Three of *The Geometry* seems to be justified. If, however, the essence of analytic geometry, and in general of "modern mathematics", consists mainly in the new proposal for the solution of the two thousand-year old question regarding the magnitude-number relation, then also the first book of *The Geometry* reveals itself significant for the development of modern mathematics[3]. Indeed, it was there that the mathematical tools were set forth which then seemed appropriate for solving the problem of the magnitude-number relation, and thus also the problem of the continuous-discrete relation, thanks to the introduction of the concept of the "unit segment" (a line segment of an arbitrarily chosen length, determined as a "unit", and meant to correspond to "1" in arithmetic). After having chosen a fixed line conceived as a "unity" Descartes could define operations on line segments corresponding to the arithmetical operations. The subsequent "merging" of the geometrical problems into algebraic formulas on the one hand, and on the other the institution of the Newtonian concept of the number understood as a quantity related to a unity (and ultimately, it seems, to a "unit segment"), was but a consequence of this first step[4].

According to Descartes himself, topics considered by him in the first book of *The Geometry* aimed at the construction of the *mathesis universalis.* As for the five arithmetical operations, referred to at the beginning of this book, and performed with lines related to a "unit segment", they signal, simple as they are, modification of the foundations of mathematics. In fact, since the Eudoxus' reply to the question of incommensurable magnitudes, in the 5th century B.C., connecting "ratio" and "proportio" to geometry, (*Elements* V), the attribution of n u m e r i c a l values to the l e n g t h s o f l i n e s e g m e n t s (and to other quantities, such as for instance magnitudes of angles), had been excluded[5]. This situation resulted from a hiatus between m a g n i t u d e and n u m b e r (between geometry and arithmetic), due, ultimately, to the concept of number being limited to the integers alone, and thus inapt to express incommensurabilities. The new formalism, founded with the introduction of the concept of the "unit segment" of a line into mathematics, embraced equally numbers and magnitudes, and made it possible to

bridge the gap between "continuous" and "discrete"[6]. Thus, the geometrical concept of the "unit of length" had an impact on the concept of the number. Once the b i u n i v o c a l correspondence between lengths and the ratios of magnitudes was established, the subsequent geometrical definition of arithmetical operations – assuming a fixed unit – led to the geometrical definition of the field of the real numbers: the expression of magnitude through the measure of length (Bianchini, Bombelli, Stevin, Viète, Descartes). Consequently, the ancient geometrical mathematics could be abandoned, which led, when used by "naïve" calculators, to such logical inconsistencies as operations on different mathematical entities: lines, planes and solids (contrary to the requirements of the famous "principle of homogeneity")[7]. Furthermore, the use of the "unit segment", and thus the quantification of magnitudes (the expression of the ratio of magnitudes through lines whose lengths accorded with the arbitrarily preestablished unit segment), influenced the structure of mathematics: it led to the reversal of the relation between algebra and geometry.

It is a truism to say that the Cartesian mathematics had an impact upon early modern physics, providing it with a tool proper to a quantitative descrtiption of nature. Newtons *Philosophiae naturalis principia mathematica* and the idea of the absolute space included in them was but one of the results obtained with this tool[8].

Until quite recently the "invention" of the "unit segment" has been commonly attributed to the seventeenth-century mathematicians. The fact, however, that Descartes was the first to publish the complete and coherent exposition of the function of the "unit segment" in geometry does not mean that he was the first to see the advantages of the device. As it is known, in the mathematics of Islam the constructions with the "unit segment" date from at least the 9th century. Equally al-Farabi (870–950) Thebit (830–901) al-Bagdadi (d. 1037) and Omar Khayyam (1048–1130) used them as a remedy against the formal inconsistencies that resulted from arithmetical operations with "heterogenous" geometrical objects[9]. Also Descartes' choice of constructions involving the "unit segment" aimed at avoiding these inconsistencies. However, there exists a significant nuance between the Muslims' proposal and Descartes' one. Contrary to the Oriental tradition that introduced not only unitary lines into arithmetical operations, but also unitary planes and solids (as did Bianchini in the fifteenth century Italy), Descartes used a "unit of length" to express all dimensions. In this way he liberated mathematics from the "principle of homogeneity", proposing instead the "principle of nonhomogeneity", based on the following proportion:

$$1 : a = a : a^2 = a^2 : a^3 ...$$

Thus the p r o d u c t ab=c is defined by Descartes as the p r o p o r t i o n:

$$"unit segment" : a = b : c$$

the procedure that will be subsequently adopted by Leibniz and Newton.

In Descartes' words:

> "Il est aussy a remarquer que toutes les parties d'vne mesme ligne, se doivent ordinairement exprimer par autant de dimensions l'une que l'autre, lorsque l'vnité n'est point determinée [...] mais que ce n'est pas de mesme lorsque l'unité est déterminée, a cause quelle peut estre sousentendue par tout ou il y a trop ou trop peu de dimensions". [p. 299]

1. ITALIAN RENAISSANCE MATHEMATICS AND DESCARTES' ACHIEVEMENT

Studies on the Italian Renaissance mathematics that were undertaken at the beginning of the 20th century revealed much the same function of the "unit segment", as the one that occurs in Descartes' *La Géometrie*, in Raphael Bombelli's *L'Algebra. Parte magiore dell'arithmetica*, a book written about 1560 and printed in 1572. Bombelli's doctrine about the "unit segment" is still present in another work by him, which is in fact a continuation of *L'Algebra...*, and contains books IV and V of it. The later work, preserved in a manuscript copy in the Biblioteca dell'Archiginnasio, Bologna, ms. nr. 1569, and composed about the same time as *L'Algebra...*, was discovered as late as 1923 and first published in 1929[10]. The assimilation of all measures of magnitude to the measure of length was done by Bombelli expressly. In fact he calls his exposition of algebra the *Algebra linearia*.

About a hundred years later Descartes proceeded in a similar way, when he presented the "unit segment" as a device which would enable an approach to lines in terms of numbers, and vice-versa:

> "Ainsi n'at'on autre chose a faire en Géometrie touchant les lignes qu'on cherche, pour les preparer a estre connues, que leur en adiouster d'autres, ou en oster, Oubien en ayant une, que ie nommeray l'v n i t é pour la r a-p p o r t e r d'autant mieux aux n o m b r e s, & qui peut ordinairement estre prise a d i s c r e t i o n, puis en ayant encore deux autres, en trouuer vne quatriesme, qui soit à l'vne de ces deux, comme l'autre est a l'vnité, ce qui est le mesme que la Multiplication; [...] Et ie ne craindrai pas dintroduire ces termes d'Arithmetique en la Geometrie, affin de me rendre plus intelligible." [p. 297–298]

Recently, one more pre-Cartesian geometrical construction involving the "unit segment" was discovered, about a hundred years older than the one extant in Bombelli's works (and at the present state of research, the oldest known in the Occidental mathematical tradition). The construction is included in Giovanni Bianchini's *De arithmetica*, a treatise written in Ferrara about 1440, and it was first published in 1994[11]. It is not our aim to consider here possible Bombelli's dependence on Bianchini. It seems significant, however, that Bianchini was

renowned by sixteenth-century mathematicians. Cardano, for instance, commented on Bianchini's tables devoted to spherical trigonometry (*Tabulae primi mobilis*)[12].

Bianchini's and Bombelli's treatises seem to point to the "European story" of the pre-Cartesian constructions with the "unit segment". It is not inconceivable that this story was rooted in the mediaeval Oriental mathematics.

2. GIOVANNI BIANCHINI'S *DE ARITHMETICA*

Bianchini is a personality of the fifteenth century[13]. His dated works come from the decades between 1442 and 1466. Presumably, however, he was ready to undertake research on mathematics as early as the twenties of the fifteenth century, when he left Venice for Ferrara, invited there in 1427 by Niccolò d'Este for the purpose of taking care of the finances of the d'Este's Court. The duty required an accomplished mathematician. As a rule, such mathematicians were educated at the *scuole d'abaco*. According, however, to Bianchini himself, he was an autodidact. The fact that he had been interested in mathematics and astronomy since his early youth spent in Venice might be pertinent to the question of possible Oriental inspirations of Bianchinis doctrines, since Venice at that time was a place of not only commercial exchanges.

One of Bianchinis earliest dated works is the *Compositio instrumenti* (1442), a treatise on the construction and use of a surveying instrument, famous thanks to the decimal positional fractions that were introduced in it for the sake of computational purposes, together with an explanation of the use of the decimal point[14]. The correspondence with Regiomontanus is among Bianchini's last dated works. It began in 1463 and lasted for some months[15]. In the period of about twenty years that separated these two works Bianchini dressed several sets of astronomical tables, and of the tables of trigonometric functions that he accompanied by rules of their use (*Canones tabularum*). The astronomical *summa* entitled *Flores Almagesti* occupies a particular position among Bianchini's works as well as among the entire fifteenth-century scientific production[16].

The *Flores*, inspired by Ptolemy's *Almagest,* and composed of ten copious treatises, were intended by their author to provide the reader with the whole body of astronomy. The mathematical introduction to the *Flores* comprises the treatises: *De arithmetica*, *Algebra*, and *De proportionibus*, followed by an exposition of the elements of plane and spherical trigonometry that also touched upon the composition of the tables of trigonometrical functions, and their use for the solution of problems of spherical trigonometry. It is the *De arithmetica* that is particularly pertinent to the present study[17].

The *De arithmetica* belongs to a small group of the fifteenth-century arithmetical treatises intended for university milieus. Composed of twenty one chapters

written in Latin, it differs from fifteenth-century treatises devoted to commercial arithmetic, written mostly in vernaculars[18]. Furthermore, Bianchini's ambition, at variance with those of the writers of the "trattati d'abbaco", was to present not only rules of calculation, but also to give geometrical principles on which they were based.

In the *Introduction* to the *De arithmetica* Bianchini presents the decimal positional number system. From the *Introduction*, and then from the chapters devoted to the five arithmetical operations, emerges Bianchini's concept of number. This concept is confirmed subsequently by Bianchini's doctrine on proportions to be found in the *De proportionibus*, and then by his exposition of arithmetical operations with decimal positional fractions, found in the *Compositio instrumenti*. In the further parts of my paper I will refer to the *De arithmetica* and *De proportionibus* (i.e. to the first and third treatises of the *Flores Almagesti*) as well as to the *Compositio instrumenti*. All these writings, coming from the early forties of the fifteenth century, are preserved in manuscripts from the period included between the mid-fifteenth and the first decade of the sixteenth century.

3. A NUMBER AS AN EXPRESSION OF A QUANTITY. "NUMERALS" AND "NUMBERS". THE "ONE" CONSIDERED AS A NUMBER AMONG OTHER "SIMPLE NUMBERS"

The following "definitions" of the number open the *De arithmetica*:

"I state that arithmetic is determined by numbers. I note two definitions of them, viz. definition of the simple number and of the composed one. And generally each number, simple as well as composed, of any quantity: little, large or largest, consists of only nine substantial numerals. The denominations are given to numbers accordingly to the order of places occupied by the numerals"[19].

The two "orders of numbers" are discussed in the next section of the *Introduction*:

"First [I will tell] of the first order. Any simple number is included between unity and nine. [...] The numerals with their q u a n t i t y are the following: one, two, three, four, five, six, seven, eight, nine: 1 2 3 4 5 6 7 8 9. Of these numerals are composed all numbers of a n y q u a n t i t y that exists. A tenth numeral is added to them, viz. the "zero" which is called the "image of number", *cifra*. It denotes nothing in numbers.

As for the second order, by which are noted all quantities that can be expressed by simples numbers as well as by composed ones, [it includes]: numbers, decades, hundreds, thousands and millions, as called by the Latins"[20].

According to Bianchini:
- the substantial numerals are nine, to which, the zero is added,
- since each of numerals denotes a respective simple number thus the simple numbers are also nine (**the unity is considered as a number**),
- each number represents a q u a n t i t y that corresponds to this number.

Thus, the essence of Bianchinis number concept is the number-quantity relation. Incidentally, it is interesting to note that Bianchini's statement evokes the well known definition of the number put forward by Simon Stevin at the beginning of the *Arithmétique* (1585):

"Nombre est cela, par lequel s'explique la quantité de chacune chose".

Actually, Bianchini anticipates Stevin's concept of number. Furthermore, at variance with the tradition established by the Pythagoreans and followed by 16th century mathematicians (for instance by Bombelli o.c., p. 11), Bianchini begins the series of numbers with "one". The "one" is a numeral among other numerals, and a number among other numbers. Numbers do n o t originate through the cumulation of the indivisible units (that are themselves not numbers). At this point, Bianchini's concept of number evokes that of Newton. I mean here the first part of the famous definition of the number given by Newton at the begining of the *Arithmetica universalis* (1707), in which it is explained what the number is n o t:

"Per numerum non tam multitudinem unitatum [...] intelligimus..."

4. THE NUMBER "ONE" IN THE ARITHMETICAL OPERATIONS

Since the "one" is considered by Bianchini as a number (and not as a special entity such as a "principle generating numbers"), thus it can be manipulated in arithmetical operations in the same way as the other numbers are. For instance, being divisible itself, it may also function as a divisor. Although the possibility of division by one is not stated explicitly by Bianchini (perhaps just because of the self-evident nature of this operation), it emerges from the rules of division of fractions and division by fractions, given by Bianchini in the *Introduction* to the *De arithmetica*. In fact, when considering the division of fractions by integers or by fractions, or else when considering the division of integers by fractions, Bianchini uses a concept of the "inverse of number" for multiplication: one divided by an integer or by a f r a c t i o n (fractions are considered by Bianchini to be numbers).

"Regulae" given in the *Introduction* to the *De arithmetica*:

"When integers are divided by fractions they are multiplied by unity in proportion to the divisor.
When fractions are divided by integers they are multiplied by unity in proportion to the number divisor.

> When fractions are divided by fractions the dividend is multiplied by
> unity in proportion to fractions of the divisor"[21].

Bianchini seems to be the first in the West to procede in that manner when fractions are involved in the division (*i.e.* to apply the inverse of the divisor for multiplication). This procedure seems to be simply a consequence of his concept of number. According to the "definitions of numbers" and the "Rules of operations with fractions" Bianchini's "number" results from the relation of the quantity to its (preestablished) unit. Consequently, the multiplication of numbers may be expressed through a proportion $1 : a = b : ab$.

Thus, contrary to the earlier mathematicians for whom the "ratio" of two quantities (or of two numbers) expressed their commensurability, for Bianchini the "ratio", involved in the concept of number (in Bianchini's concept of number fractions are included), expresses the commensurability of a quantity with the unit of this quantity.

Newton's definition of number (the beginning of which was quoted above), is equally based on the relation of a quantity to its determined unit.

> "Per numerum non tam multitudinem unitatum, quam abstractam quan-
> titatem cujusvis ad aliam ejusdem generis quantitatem, quae pro u n i t a t e
> habetur, rationem intelligimus." [the beginning of the *Arithmetica univer-
> salis*].

Newton, however, formulates this relation overtly, unlike Bianchini whose idea of the relation "quantity – its unit" results from the rules of the operations with fractions.

5. BIANCHINIS CONCEPT OF THE "UNIT SEGMENT" AND THE INFINITE (?) DIVISIBILITY OF THE UNITY: THE *INTRODUCTION*, CHAPTER VIII, AND THE BEGINNING OF CHAPTER XII OF THE *DE ARITHMETICA*, AS WELL AS CHAPTERS II–IV OF THE *COMPOSITIO INSTRUMENTI*

The problem of the infinite divisibility of the unity appears in Bianchini's *De arithmetica* and in the *Compositio instrumenti* on three occasions, namely: when the definitions of number are formulated, then when the concept of the common fractions and of operations with them is treated, and finally when the concept of the decimal positional fractions and of operations with them is introduced.

In the *De arithmetica* two of these subjects are present (although the extension of the decimal number system to fractions is only briefly touched upon), whereas in the *Compositio instrumenti* attention is paid to the explanation of the principles of the decimal positional fractions, together with rules of operations with them, the use of the decimal point included[22].

Although it is difficult to establish which of these two treatises was written first, I suppose that the *De arithmetica* had been written before the *Compositio instrumenti*[23]. In the *De arithmetica* Bianchini considers, apart from the subjects mentioned above, subtractive and negative numbers, quadratic irrationals, and surds in general.

In chapter VIII of the *De arithmetica*, entitled *De practica in radicibus universalibus operanda*, Bianchini states

> "Root" has the same meaning as "principle" or "beginning" or "base", and it acquires its name [value] in accordance to its determined end. Sometimes this "determined end" is looked for on the basis of the known root, and sometimes on the contrary, by a given name [value] the root is searched for, from which it originated.
>
> Sometimes the name [value] of the root is given in numbers, and also the root [of this "value"] is given in discrete numbers.
>
> Sometimes, however, it is not possible to find out a root in discrete numbers, and then it is necessary to find it in a continuous quantity, and this sort of root is usually called a "surd root". It is found out through the geometrical demonstrations, by l i n e s or p l a n e s or s o l i d s"[24].

Bianchinis doctrine on surds is completed in Chapter XII of the *De arithmetica*: *De radicibus surdis in quantitate continua inveniendis*. This chapter, that subsequently will be the subject of our particular attention, begins as follows

> "I want a square root of 24. And since it is not possible to find it in numbers, I am searching for it in continuous quantity which means demonstrating it [its value] through a line.
>
> And for the sake of the explanation of the reason why this root can not be found in integers, I state that this is obvious because the value of this root has to be more than 4 and less than 5, and what happens between 4 and 5 are fractions.
>
> And it is known by the the third definition of this [treatise] that any fraction multiplied by itself i n c r e a s e s i n f r a c t i o n and escapes from the unity, and will never produce an integer"[25].

This "third definition" to which Bianchini refers is as follows

> "When fractions are multiplied by fractions the product will be fractions of fractions"[26].

Bianchini had two options of coping with fractions or surds considered as numbers: either to introduce into arithmetic concepts proper to geometry (which would lead him to a logical inconsistency), or to make recourse to proportions. He chose the second. Once the legality of division of "one" and by "one" was established, which means also: once it was legitimate to introduce the "one" to the theory of proportions (according to Bianchini proportions are fractions), Bianchini could do both: first to apply the "one" to operations with fractions (in fact, the division of a fraction by an integer or by a fraction is intended by Bianchini as

a multiplication of this fraction by the inverse of the divisor), and second to express the number "one" geometrically, *i.e.* by a "unit segment", and to apply it to the arithmetical operations with surds (extraction of a surd root).

6. DESCARTES, BOMBELLI AND BIANCHINI: ON THE APPLICATION OF THE "UNIT SEGMENT" TO THE EXTRACTION OF SQUARE ROOTS (FROM NOT SQUARE NUMBERS).

Since the idea of the application of the "unit segment" to the algebrical transformations is latent in the *Elements* it is thus to the *Elements* that Bianchini refers himself (following in this way the example of the Muslim mathematicians or independent of them?). As for Bombelli and Descartes, both of whom had the same background in geometry as Bianchini, neither of them refers to Euclide or considers it necessary to give proof of their statements concerning the use of the "unit segment"[27].

The famous fragment of *The Geometry* runs as follows

> "...Ainsì na-t-on autre chose a faire, en geometrie, touchant les lignes qu'on cherche, pour les preparer a estre connues, que leur en adiouster d'autres;
>
> ou bien, en ayant vne que ie nommeay l ' u n i t é pour la rapporter d'autant mieux aux nombres, et qui peut ordinairement estre prise a discretion, puis en ayant encore deux autres, en trouver vne quatrieme, qui soit a l'vne de ces deux comme l'autre est a l ' u n i t é ce qui est le mesme que la Multiplication; [...]
>
> ou enfin trouver une, ou deux, ou plusieurs moyennes proportionnelles entre l ' u n i t é et quelque autre ligne, ce qui est le mesme que tirer la racine carrée, ou cubique, etc." [p. 297–298]

Then the exemplification of the principle is given through the case of the extraction of a square root

> "Ou s'il faut tirer la racine quarrée de GH, ie luy adiouste en ligne droite FG, qui est l'vnité, & diuisant FH en deux parties esgales au point K, du centre K ie tire le cercle FIH, puis esleuant du poit G vne ligne droite iusques à I, a angles droits sur FH, c'est GI la racine cherchée.
>
> Ie ne dis rien icy de la racine cubique, ny des autres, à cause que i'en parleray plus commodement cy aprés." [p. 298]

The same idea is expressed in the following way in Bombellis *L'Algebra*...:

> "Sia linea b d la quale sia 7, cioè sette volte linea g, e che si detta linea se ne voglia il creatore. Allunghisi la d b sino in a, et sia la a b pari a la g, et sopra la a d si faccia il semicirculo a f d, e dal punto b si tiri ad angolo retto la b f sino che tocchi la circumferentia f, la b f sara il creatore della b d, cioè di 7."[28]

As for Bianchini's idea of the application of the "unit segment" to the extraction of the square roots from not square numbers, it is presented in chapter XII of the *De arithmetica*, entitled *De radicibus surdis in quantitate continua inveniendis.* Bianchini's chain of reasoning, reconstructed above thanks to the elements of Bianchini's teaching on mathematics inherent in the *De arithmetica* and *De proportionibus* just leads to the concept of the mean proportional, in which the "unit segment" is involved. Then, in the 16th and 17th centuries, the same concept of the "unit segment" will function in Bombelli's *Algebra linearia* and in Descartes' *La Géometrie*.
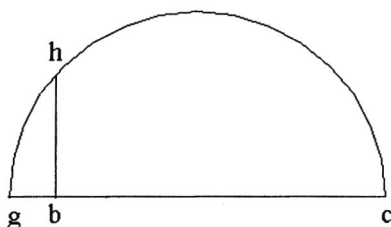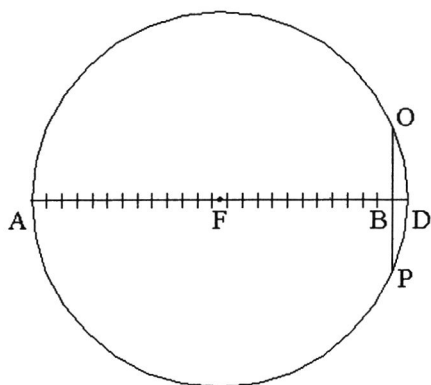
Bianchini: Bombelli:

De radicibus surdis...   Modo di trovere il lato di un numero in linea.

Quaero radicem de 24. Et quia in numeris non est posibile invenire ipsam, quaero in quantitate continua, scilicet in linea demonstrare ipsam.[...]

Est enim linea longitudinis 24 producta secundum mensuram tuam, sit ergo linea AB.

Sia linea *a* una misura data per la unita [...] e la linea *b c* si è 7 delle dette misure, della quale si voglia il lato,

Et continuabo [eam] a puncto B per quantitatem u n i u s numeri, secundum mensuram primo mensuratam, quae sit BD.

allunghisi *c b* fino in *g* facendo *b g* pari all'*a*,

Deinde totam lineam AD dividam in duabus partibus aequalibus in puncto F, super quem firmabo pedem circini et componam circulum secundum quantitatem diamtri AFBD.

e sopra la *c g* si faccia mezo cerchio *c h g*,

Postmodum supra punctum B erigam perpendicularem usque ad circumferentiam, quae sit BO, quam etiam continuabo usque in P, et erit linea OBP.

e del punto *b* si tiri *b h* ad angolo retto sopra la *c g* fino alla circonferentia *h*;

Dico enim quod linea BO est latus quadrati supeficiei de 24.

la *b h* sara il lato di *b c* cioè di 7.

Et hoc probatur per 14 secundi atque per 3 et 34 tertii [Euclidis...].

CONCLUSIONS

Geometrical constructions with the "unit segment" developed between the 9th and 17th centuries. They were introduced in the Moslem East as early as the 9th century and developed until 13th. Subsequently, in the first half of the 15th century, they appeared in the West, in Italy, in the mathematical Introduction to an astronomical *summa*. Although the author of this *summa* was esteemed by his contemporaries for mathematical and astronomical skills, and his work known throughout Europe, the constructions involving the "unit segment" did not exert any influence, it seems, during the next hundred years. The next time the "unit segment" appears is about 1550, in Rafael Bombelli's treatises on algebra (published in 1572 and in 1929), then it is to be found in Simon Stevin's *Arithmétique* (1585), and finally in Descartes *La Géometrie* (1637). The constructions with the "unit segment" contributed to the modern concept of the number given in Newton's *Arithmetica universalis* (1707).

In Bianchini's *De arithmetica* the "unit segment" is present as a consequence of Bianchini's own concept of unity, and of the role of the unity in proportions. But, albeit this "non-accidental" presence, it does seem to be used by Bianchini casually, and to function merely as an illustration (a very proper one) of the magnitude-number relation; or else as an indication of the possibility of the "quantification of magnitude" (which is not of minor importance), without, however, indicating all its power as a mathematical tool. As for Bombelli, he defines arithmetical operations on lines, and thus he builds his *Algebra linearia* on the concept of the "unit segment". Finally, Descartes introduces the concept of the "unit segment" systematically into mathematics, with a view to constructing of the *mathesis universalis*.

At this point several questions pose themselves concerning, *i.a.*, the possible ways of transmission of the concept of the "unit segment" from Oriental mathematics to the West (via Venice ?), and subsequently of the possible influence of Bianchini's *De arithmetica* upon Bombelli's works, and thus upon the develop-

ment of the early modern mathematics. Further research is required in order to answer these questions.

Notes

[1] H. W e y l : *Das Kontinuum*. Leipzig 1918, p. 16. Reprinted in 1932. The English translation H. W e y l : *The Continuum. A Critical Examination of the Foundation of Analysis*. Transl. by S. P o l l a r d & T. B o l e . New York 1994 Dover Publications, Inc. pp. 23–24. See J. D h o m b r e s : *Nombre, mesure et continu. Epistémologie et histoire*. Paris 1978 CEDIC, pp. 50–65.

[2] "La façon de distinguer toutes les lignes courbes en certains genres. Et de connoistre le rapport quont tous leurs poins a ceux des lignes droites...". I refer myself to the first edition of the *La Géometrie* (1637), and I give the original ortograph of the title of the book and of the quoted fragments. The English translation is by D.E. S m i t h and M.L. L a - t h a m : "The method of distinguishing all curved lines of certain classes [...] I think the best way to group together all such curves and then classify them in order, is by recognizing the fact that all points of those curves which we may call "geometric" [...] must bear a de- finite relation to all points of a straight line, and that this relation must be expressesed by means of a single equation". *The Geometry of René Descartes*. New York 1954, Dover Publications, pp.viii and 48. This does not mean, however, that Descartes employed consequently two coordinates. The problem of coordinates used by Descartes is discussed by F. C a j o r i : *A History of Mathematics*. Second Edition, Revised and Enlarged. New York 1955 (ninth printing), p. 175. See also C.H. E d w a r d s, Jr. : *The Historical Development of the Calculus*. New York, Heidelberg, Berlin 1979, Springer-Verlag, p.95.

[3] N. B o u r b a k i : *Éléments d'histoire des mathématiques*, Nouvelle édition revue, corrigée et augmentée, 1974, Paris, Hermann, p.35.

[4] It is generally admitted by historians of the calculus that in the early developments of analytic geometry the unknowns were considered line segments rather than numbers. See for instance C.H. E d w a r d s Jr , l.c. and C.B. B o y e r : *Descartes and the geometrization of algebra*. "American mathematical monthly". Vol. 66 1959 pp. 390–393. J.M.H. B o s : *Algebra and geometry around 1600*. In: "XIXth International Congress of History of Science. 22–29 August 1993 Zaragoza (Spain). *Symposia Survey Papers. Plenary Lectures*. J. D h o m b r e s, M. H o r m i g o n & E. A u s e j o (eds.), pp. 54–58.

[5] Ian M u e l l e r : *Homogeneity in Eudoxus's Theory of Proportion*. In: "Archive for the History of Exact Sciences". Vol. 7 1970/1971 pp. 1–6.

[6] D e s c a r t e s: *Regulae ad directionem ingenii*. I follow the translation in N.F. S m i t h , *Descartes: Philosophical Writings*. New York 1953, St. Martins Press. "The Rule IV: [...] There is now flourishing a certain kind of arithmetic, called algebra, which endeavors to accomplish in regard to numbers what the ancients achieved in respect to geometrical figures. These two sciences are no other than spontaneous fruits originating from the innate principles of the method in question.

The Rule XVI: [...] What above all requires to be noted is that the root, the square, the cube, etc., are merely m a g n i t u d e s   in   c o n t i n u e d   p r o p o r t i o n, which always implies the freely chosen unit [...]. The first proportional is related to this   u n i t   immediately or by one single relation, the second by the mediation of the first and the second, and so by three relations, etc. We therefore entitle the   m a g n i t u d e, which in algebra is called the   r o o t, the   f i r s t   p r o p o r t i o n a l; that called the   s q u a r e   we shall speak of as being the   s e c o n d   p r o p o r t i o n a l, and similarly in the case of the other." (The *spatia* are mine G.R.).

[7] As it is noted by E. G i u s t i , the principle of homogeneity caused problems to Viète. He avoided them by stating the solution of equations *using proportions between the unknown and the given magnitudes, on the base of the Elements VI, 16.* E. G i u s t i: *Algebra and Geometry in Bombelli and Viète.* "Bollettino di storia delle scienze matematiche". Vol. 12, 1992 pp.319–320.

[8] R. D u d a : *Newton and the Mathematical Concept of Space.* In: *Issac Newton's "Philosophiae naturalis principia mathematica".* W.A. Kamiński (Ed.): World Scientific. Singapore, New Jersey, Hong Kong 1988, pp. 72–83.

[9] G.P. M a t v i e v s k a y a , *Učenje o čisle ...* Tashkent 1967 pp. 217–218, 233, 260–261.

[10] E. B o r t o l o t t i : *L'Algebra. Opera di Rafael Bombelli da Bologna.* Libri IV e V comprendenti «la parte geometrica» inedita tratta dal manoscritto B. 1569 della Biblioteca dell'Archiginnasio di Bologna. N. Zanichelli, Bologna 1929. 2nd edition, together with books I–III. Prima edizione integrale. Milano 1966. I refer myself to the 2nd edition.

[11] G. R o s i ń s k a : *A chapter in the history of the Renaissance mathematics: negative numbers and the formulation of the law of signs (Ferrara, Italy ca.1450).* „Kwartalnik Historii Nauki i Techniki" ("Quarterly Journal of the History of Science and Technology"). Vol. 40 1995 nr 1 p.15, note 13. I would like to correct at this occasion an error in print, namely instead of *Elements* II, 24 it should be *Elements* II, 14, thus instead of: [...] probatur per 24 secundi [...] it should be: probatur per 14 secundi.

[12] H i e r o n i m i   C a r d a n i Medici Mediolanensis: *Libelli quinque... De supplemento Almanach... De restitutione temporum...* Norimbergae, 1547 apud Johannem Petreium, on f.183r *Canon pro tabulis Blanchini sequentibus:* Hic sunt duae tabulae pro nostra regione, una ascensionum rectarum stellae habentis latitudinem non maiorem octo partibus, altera obliquarum earundem... ...potest etiam eximere nostram obliquam et loco eius propriam reponere.

[13] G. F e d e r i c i   V e s c o v i n i : *Bianchini, Giovanni (Iohannes Bianchinus. Iohannes de Bianchinis).* "Dizionario biografico degli Italiani". Vol. 10 1968 pp.194–196.

[14] G. R o s i ń s k a: *Decimal positional fractions. Their use for the surveying purposes (Ferrara, 1442).* „Kwartalnik Historii Nauki i Techniki". Vol. 40 1995 nr 4 pp.17–31. Bianchini's treatise *Compositio instrumenti* was published by P. G a r u t i (with an historical Introduction by G. A r r i g h i): *Giovanni Bianchini. Compositio instrumenti* (Cod. Lat. 145 T.6.19) della Biblioteca Estense di Modena. "Rendiconti Classe di Lettere e Scienze Morali e Storiche". Vol.125 no 1 1991 pp.95–127. (Istituto Lombardo Accademia di Scienze e Lettere, Milano 1992).

e Scienze Morali e Storiche". Vol.125 no 1 1991 pp.95–127. (Istituto Lombardo Accademia di Scienze e Lettere, Milano 1992).

[15] See G. F e d e r i c i V e s c o v i n i , op. cit. and A. G e r l : *Trigonometrisch-astronomisches Rechnen kurz vor Copernicus. Der Briefwechsel Regiomontanus-Bianchini*. Stuttgart 1989. (Boethius, vol.XXI).

[16] I am preparing a study on Bianchinis *Flores Almagesti*, in which the audiance of Bianchinis mathematical and astronomical works in the 15th and 16th centuries is considered. I signal the dependence of Regiomontanus *Tabulae directionum* on Bianchinis *Tabulae primi mobilis* in the paper *Giovanni Bianchini – matematyk i astronom XV wieku (Giovanni Bianchini: fifteenth-century mathematician and astronomer)*. In: „Kwartalnik Historii Nauki i Techniki". Vol. 26 1981, English summary on p.577. In this paper the results of the comparison of the first edition of Regiomontanus *Tabule directionum* with the copy of Bianchini's *Tabulae primi mobilis* done in Rome about 1464 and preserved in the Jagellonian Library, Cracow (Biblioteka Jagiellońska) ms. BJ 556, are given. In fact, Regiomontanus' *Tabula ascensionum rectarum* is an almost exact copy of Bianchinis table similarly headed, dressed in 1456 (ms. BJ 556, ff.66r–92v). The rubric *equacio dierum,* however, present in Bianchini's table, is missing in Regiomontanus' *Tabula ascensionum rectarum*. Then, Bianchini's table headed *Ascensiones signorum in circulo obliquo,* dressed for geographical latitudes 35–54, was rearenged by Regiomontanus and presented by him as two separate tables, headed T*abula ascensionum obliquarum* and *Tabule celi mediationum*. Finally, Regiomontanus' *Tabula declinacionis generalis* is an exact copy of Bianchini's *Tabula novissima declinacionis ecliptice per arcum*.

Copernicus, while studying in Cracow, inscribed in his note-book, preserved till now in Uppsala (Copernicana 4), his-own version of Bianchini's tables of the movement of the planets in the latitude. G. R o s i ń s k a : *Identyfikacja szkolnych tablic astronomicznych Kopernika (Identification of Copernican Tables of the latitude of planet....)* „Kwartalnik Historii Nauki i Techniki". Vol. 29: 1984, pp.637–644 (in Polish with an English Summary).

The possible influence of Bianchini's *Algebra* upon the teaching of mathematics at universities of Central Europe is considered in my paper *The channels of transmission of the fifteenth-century algebra from Italy to Central Europe. The case of Giovanni Bianchini's "Flores Almagesti"*. „Organon". Vol. 26 1997 (forthcomming).

[17] All quotations of the *De arithmetica* come from the critical edition of this treatise that I have prepared for the *Studia Copernicana* series. The edition is based on six extant manuscript copies of the *De arithmetica*, preserved in the following libraries: Bologna, Biblioteca Universitaria, ms. 19(23), ff.1r–13v. Cracow, Biblioteca Jagiellońska, ms. 558, f.1r–12r. Paris, Bibliothèque Nationale, ms. 10253, f.6r–23r. Perugia, Biblioteca Comunale Augusta, ms. 1004, f.1r–8r. Città del Vaticano, mss. Vat. Lat. 2228, ff.16–25v and Vat. Reg. 1915, ff.38r–52r. I adopt the Renaissance spelling of the Latin text. Some elements of this spelling are present in the text preserved in the ms. Bologna 19(23).

[18] W. van E g m o n d : *Practical Mathematics in the Italian Renaissance: A Catalog of Italian Abbacus Manuscripts and Printed Books to 1600.* "Supplemento agli Annali dell'Istituto di Storia delle Scienze di Firenze" fasc.1. Firenze 1980.

[19] Johannis Blanchini *Florum Almagesti tractatus primus*. De arithmetica. Prohemium. Arithmetica dico quod determinatur per numeros. Et duas noto definitiones numerorum, videlicet numerus simplex et numerus compositus. Et generaliter omnis numerus, tam simplex quam compositus, cuiuscumque quantitatis, parvae, magnae seu maximae, componitur solummodo cum novem substantialibus figuris [...]

[20] Et primo de ordine primo. Dico, quod numerus simplex comprehenditur solummodo ab unitate usque in novem et cum novem figuris demonstratur quilibet secundum suam quantitatem. Quae figurae cum cuiuslibet quantitate sunt unum, duo, tres, quattuor, quinque, sex, septem, octo, novem: 1 2 3 4 5 6 7 8 9.

Et in istis figuris consistit omnis numerus cuiuscumque quantitatis existat. Quibus etiam addita est decima figura, videlicet 0, quae vocatur figura numeri, cifra, et nihil in numeris denotat.

In secundo autem ordine, per quem notantur omnes quantitates, tam simplices quam compositae quae numerari possint [quantitates istae] determinatae sunt per quinque denominationes secundum Latinos, videlicet: numeri, decenae, centenaia, milliaria et millioni.

[21] *De arithmetica*, Prohemium [Regulae:] [...] Quando integri dividuntur per fractiones multiplicentur integri secundum proportionem unitatis ad ilasmet fractiones divisoris.

Qando fractiones dividuntur per integra multiplicentur fractiones secundum proportionem unitatis ad numerum divisorem.

Quando fractiones per fractiones dividuntur multiplicentur fractiones dividendae secundum proportionem unitatis ad fractionem divisoris.

[22] G. R o s i ń s k a : *Decimal positional fractions...* op. cit., pp.24–25.

[23] I consider this question in the *Decimal positional fractions...* op. cit., p.17.

[24] *De arithmetica*, c.8: Radix idem sonat sicut principium vel ortus aut fundamentum et secundum eius determinatam finem acquirit pronomen. Et aliquando per notam radicem datam quaeritur eius determinatus finis et aliquando econverso, per datum pronomen quaeritur radix ex qua oritur. Quod aliquando datum est pronomen radicis in numeris etiam ipsius radix in numeris reperitur discretis, et aliquando non est possibile ipsam invenire in numeris discretis, et tunc necesse est ipsam invenire in quantitate continua. Et haec vocatur communiter *surda* radix et invenitur per lineas et superficies aut corpora cum geometricis demonstrationibus.

[25] *De arithmetica*, c.12: Quaero radicem quadratam de 24. Et quia in numeris non est possibile ipsam invenire, quaero in quantitate continua, scilicet in linea demonstrare ipsam. Et ut tibi patefiat causa propter quod in numeris non invenitur, dico quod manifestum est radix ipsa in numeris integris non cadit, quia numerum ipsum oportet esse plus 4 et minus 5, et inter 4 et 5 cadunt fractiones. Et notum est per tertiam definitionem huius quod quaelibet fractio per seipsam multiplicata crescit in fractione et ab unitate elongatur, quia non unquam producitur integrum, ergo etc.

[26] *De arithmetica*, Prohemium: Quando fractiones multiplicantur per fractiones productus erit fractiones fractionum.

[27] Bombelli refers himself to the *Elements* in the first book of the *L'Algebra...*, as for Descartes relation to the *Elements* see the opinion presented by J.M.H. B o s : *The structure*

of Descartes' "Géométrie". In: J.M.H. B o s : *Lectures in the History of Mathematics.* Providence 1993, pp.37–57.

[28] *L'Algebra. Opera di Rafael Bombelli da Bologna...* op. cit. p.41. In the foot-note 23 E. B o r t o l o t t i states: "Questo e un chiaro esempio di applicazione del segmento unitario a una costruzione geometrica. Le costruzioni sono basate sui noti teoremi euclidei".

Grażyna Rosińska

MATEMATYKA XV WIEKU A POCZĄTKI MATEMATYKI NOWOŻYTNEJ
ODCINEK JEDNOSTKOWY I JEGO FUNKCJA W BIANCHINIEGO *DE ARITHMETICA*, BOMBELLEGO *L'ALGEBRA...* I KARTEZJUSZA *LA GÉOMETRIE*

Funkcja „odcinka jednostkowego" w *Geometrii* Kartezjusza jest dobrze znana: początek księgi pierwszej tego dziełka, uznanego za przełomowe w rozwoju matematyki, przynosi objaśnienie w jaki sposób, stosując taki właśnie odcinek, wyrazić geometrycznie działania arytmetyczne zarówno na liczbach wymiernych jak niewymiernych. A więc i odwrotnie – jak przy pomocy l i c z b y , będącej przedmiotem arytmetyki, wyrazić w i e l k o ś ć będącą przedmiotem geometrii. W historii matematyki odnotowuje się to, wraz z wprowadzeniem systemu współrzędnych (przez Kartezjusza oraz Fermata), jako wydarzenie otwierające drogę matematyce nowożytnej.

W ten sposób już pierwsze sformułowania w *Geometrii* Kartezjusza przynoszą, w postaci „odcinka jednostkowego", narzędzie do skonstruowania „mostu" nad przepaścią dzielącą, zgodnie z matematyką starożytnych, liczbę i wielkość. Przepaść zaś wynikła stąd, iż liczba, według koncepcji Pitagorejczyków z istoty swej „dyskretna", nie mogła wyrazić wielkości, z natury ciągłej. Wprowadzenie do geometrii „odcinka jednostkowego", który miał odpowiadać jedności w arytmetyce, nazywane jest przez historyków matematyki „trikiem", który po pierwsze uwolnił algebrę geometryczną od takich absurdów jak mnożenie „brył przez odcinki", czy „powierzchni przez bryły", następnie umożliwił Kartezjuszowi skonstruowanie geometrii analitycznej, a dalej, wraz z pracami Leibniza i Newtona, powstanie analizy matematycznej, a więc matematycznego narzędzia dla nowożytnego przyrodoznawstwa.

Włoski matematyk i historyk matematyki, Ettore Bortolotti, wykazał w latach międzywojennych, że pomysł „odcinka jednostkowego" znany był przeszło sto lat wcześniej niż użył go Kartezjusz. Rafael Bombelli posłużył się nim w swojej *Algebrze* (*Algebra linearia*), dziele napisanym około połowy XVI wieku, którego część pierwsza (księgi I–III), ukazała się w roku 1572, a część druga (księgi IV i V), w roku 1929. Zarówno w części opublikowanej w XVI wieku, jak i w tej opublikowanej w wieku XX Bombelli używa „odcinka jednostkowego" do wyrażenia geometrycznie działań na liczbach (i odwrotnie do przeprowadzania działań arytmetycznych na odcinkach).

Autorka odnalazła pomysł zastosowania „odcinka jednostkowego" do wyrażenia pierwiastkowania, w analogiczny sposób jak to czynili Bombelli i Kartezjusz, w dużo

wcześniejszym dziele, a mianowicie w traktacie *De arithmetica* napisanym około 1440 roku przez matematyka działającego w Ferrarze, Giovanniego Bianchiniego (Ioannes Blanchinus, de Blanchinis). W ten sposób historia stosowania „odcinka jednostkowego" – pomysłu ważkiego w konsekwencje dla nowożytnej matematyki – została przesunięta wstecz ponad sto lat w stosunku do dzieła Bombellego, a ponad dwieście lat w stosunku do dzieła Kartezjusza. Jest to „historia europejska" odcinka jednostkowego. W świecie Islamu bowiem był on znany już we wczesnym średniowieczu, jednak nie posłużono się nim następnie do skonstruowania „nowej geometrii", jak to w Europie uczynił Kartezjusz. Autorka nie zajmuje się możliwymi wpływami myśli arabskiej na Bianchiniego, natomiast sygnalizuje możliwość kształcenia Bianchiniego w Wenecji, będącej w XV wieku centrum wymiany nie tylko handlowej, ale także intelektualnej między Europą i krajami Islamu. Generalnie, sprawa rzeczywistych wpływów myśli Islamu na rozwój europejskiej matematyki w XV wieku (pomimo formalnego odżegnywania się uczonych Renesansu od wszelkich tradycji nie-klasycznych), nie została dotąd poddana systematycznym badaniom.