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THE EUCLIDEAN *SPATIUM* IN FIFTEENTH-CENTURY MATHEMATICS

> In so far as geometry is conceived as the science of laws governing the mutual relations of practically rigid bodies in space, it is to be regarded as the oldest branch of physics. This science was able [...] to get along without the concept of space as such, the ideal corporeal forms – point, straight line, plane, length – being sufficient for its needs. On the other hand, space as a whole, as conceived by Descartes, was absolutely necessary to Newtonian physics.
>
> *A. Einstein*

INTRODUCTION

The study of the development of geometry in periods when it functionned (also) as "branch of physics", offers keys to better understanding of the 17th century phenomenon called the "scientific revolution", when the character of the relationship, existing until then between geometry and physics, changed thanks to the Cartesian, analytic approach to Euclidean geometry. Descartes' achievement, however, was preceded by almost two centuries of the "premodern" (for the lack of a better term) investigations in mathematics and astronomy. In that way, the 17th century scholars had at their disposal mathematics that were the result of a particularly intense evolution since the early decades of the 15th century; for instance, the concept of the "unit segment", applied to the geometrical expression of arithmetical operations, was known a long time before Bombelli and Descartes made their own discoveries. The tension between arithmetic (and arithmetized algebra), on the one hand, and geometry, on the other, that marked this evolution, revealed the insufficiency of Euclidean geometry (lacking in notion of the "space as a whole, as conceived by Descartes", according to Einstein), and incited the search for a remedy.

* * *

The term "space" did not exist in Greek mathematical thought. As for the Euclidean concept of what is called *spatium* in this paper, for lack of a Greek term, it points out the spatial relations between definite objects, expressed by means of relations between straight lines conceived as segments. Consequently, the Euclidean *spatium* conceived as the Euclidean straight line E_1, the Euclidean plane E_2 and the 3-dimentional Euclidean solid E_3, is a sort of metrical expression of the form of the real, three dimensional, c l o s e d space being the subject of sensual experience. In the history of mathematics the term "l'intuition spatiale" is equally used in connection with Euclidean geometry for the sake of pointing to the peculiarity of the Euclidean *spatium*[1]. In any case, *spatium* in the *Elements*, as seen from the mathematical point of view, was a consequence of an earlier development of mathematics, and resulted equally from the Pythagorean concept of number, and from the answers to difficulties raised by this concept in mathematics. It became a challenge for mathematicians in the course of time because of the formal difficulties inherent in the *Elements*, such as the existence of the 5th postulate (the parallel lines postulate), and because of problems resulting from the "dimensionality" of the mathematical entities (as inherited from the Pythagorean metrics)[2].

In this paper, the latter reason of troubles with Euclid's concept of *spatium* will be discussed. In fact, the development arithmetic and algebra in Italy in 14th and 15th cent. brought forth mathematical expressions that could hardly be interpreted by means of three dimensional Euclidean models; it led, furthermore, to the extension of the concept of number such as to embrace also the negative numbers and incommensurable ones. When "Geometers" experianced difficulties in dealing with "non-Euclidean" or "non-Pythagorean" numbers, "Arithmeticians" hurtled against the insufficiency of the concept of *spatium* offered by Euclid. In the middle of the 15th century Giovanni Bianchini was among those who found themselves in these conditions (Simon Stevin, more than a century later, experienced the same difficulties). The situation looked serious because, according to the stantards commonly accepted in classic, mediaeval and Renaissence science, what could not be proved geometrically was judged to be "not scientific". On the other hand, geometry itself was not free from evident incosistencies: there still persisted the traditional points of collision of arithmetic with geometry which resulted from the postulate of "homogeneity" of mathematical entities involved in arithmetical operations, when these operations were interpreted g e o m e t r i c a l l y (*Elements* II). Questions were then posed of "how to multiply a line times an area?" etc. As it is known, a remedy to these incosistencies, and the proposition of geometrical interpretation of the "dimensions" of polynomials involved in arithmetical operations, came as late as in the 17th century, with Descartes' *La Geometrie* and Pierre de Fermat's *Ad locos planos et solidos isagoge* and his *Novus secundarum et ulterioris ordinis radicum in Analytics usus*. Earlier, the "multidimensionality" of space or "multidimensional geometric interpretations" of algebraic expressions were proposed by Renaissance mathematicians, beginning with Luca Pacioli, Christoff Rudolff and Michael Stiffel[3].

* * *

Research on the 15th century "mathematics for astronomers", based on manuscript sources, led me to the discovery of some traces of the "prehistory" of the evolution of the Euclidean concept of space in Western mathematics, well before the activity of Pacioli began. In this paper, problems concerning the Euclidean *spatium* will be considered as they present themselves some fifty years before Pacioli's *Summa de aritmetica, algebra, proportioni e proportionalità* was composed. In what follows, I will refer to Giovanni Bianchini's treatises *Arithmetica* and *Arithmetica algebrae*, both written around 1440 in Ferrara, and both incorporated in Bianchini's astronomical work, the *Flores Almagesti*. The first of these treatises is devoted in part to theoretical arithmetic and in part to applications of its theorems to the solution of numerical problems, and the second, *Arithmetica algebrae*, known also as *De algebra*, explains to astronomers the procedures aiming to solve the six famous forms of square equations. The exposition includes the rules of operations with negative numbers, and with algebraic fractions. Both treatises have already been the subject of preliminary studies[4].

Bianchini's exposition of arithmetic is remarkable for several reasons. First, Bianchini extends the concept of number, limited in its classical form to positive integers, and he also regards as numbers fractions, surds and negative numbers – numerical expressions of negative values – and produces geometrical proofs justifying the four mathematical operations with negative numbers. Then, he exposes the idea of the "unit segment" used for the geometrical presentation of the extraction of square root (anticipating Bombelli's and Descartes' concept of the "unit segment" used for analogue purposes) and reflects upon geometrical "justification" of the existence of powers and roots of degrees higher than the third. Finally, Bianchini deals with incommensurables in the context of decimal positional fractions which he was the first in Europe to use systematically[5].

Bianchini's achievement confirms an opinion held by historians of practical arithmetic (the arithmetic of the abacists), that it was within the framework of the search for new tools, appropiate for the solution of numerical problems, that the concept of number developed in Renaissance and early modern mathematics[6]. In fact, it was arithmetic and algebra, both developed by Bianchini for astronomical purposes, that incited him to face the incosistencies in mathematics as perceived through the relationship between arithmetic and geometry.

In what follows, first I will consider Bianchini's concept of number as well as his views on the arithmetic of natural numbers (positive integers) in relation to the Euclidean *spatium* (the case of the square, cubic and "related" roots, as well as of the "regular" roots and powers of degree higher than third). Then, I will turn to Bianchini's arithmetic of substractive and negative numbers. Bianchini's explication of how to subtract a greater number from a smaller one, and subsequently his attempt to justify geometrically multiplication of binomials composed with substractive numbers – which in fact function there as the negative ones – is a good illustration of the very special place of Bianchini's achievement in the rise of early modern mathematics. (Let me signal at this point that the same geometrical construction as that of Bianchini's, justifying operations with negative numbers in accordance with the "law of signs", was subsequently given by Simon Stevin in the *Arithmétique* (1585)[7].

1. EUCLIDEAN *SPATIUM* AND BIANCHINI'S CONCEPT OF NUMBER

As I tried to demonstrate in my former papers, in the *Arithmetica*, Bianchini breaks off with the Euclidean number understood as a result of a cumulation of units (*Elements* VII.2). Consequently, he rejects the concept of unity as being not a number in the proper sense of the term but "an origin of numbers"; at variance with Bianchini, Georg Peurbach states in the *Opus Algorithmi iocundissimum* (*ca* 1450) as follows: "Unitas autem non est numerus sed principium numeri. Unde ipsa habet se in Arithmetica sicut punctum in Geometria ad magnitudinem". In Bianchini's *Arithmetica* a unity is considered as a number among other numbers: it is divisible and, of course, it may also serve as a divisor. In fact, Bianchini uses the inversion of a number for multiplication, and furthermore he introduces unity in his concept of proportion. In that way, he can express multiplication through a "ratio"[8].

> When fractions are divided by integers, they are multiplied by unity in proportion to the number of divisor. [...]
>
> In multiplication, three [elements] are required: the multiplied, then the multiplicand, and the product. The proof of this is division, because if the product is divided by the multiplied the result will be the multiplicand, and on the contrary, if [it is divided] by the multiplicand the result will be the multiplied.
>
> And this holds for the discrete quantity[9].

Accordingly, for Bianchini, multiplication is no longer a "repeated addition" or an "abridgement of addition" (as for Peurbach, he omits the definition of multiplication in the chapter of the *Opus Algorithmi* dedicated to this operation, and passes to the examples). In fact, Bianchini seems to be the first European mathematician to state the theorem of division and to use it in the definition of multiplication.

As for Bianchini's concept of negative number (a number representing a negative value), first it appears in relation with subtraction, when the subtrahend is geater than the minuend:

> [...] thus, you have subtact 55 from 50. And since the subtrahend is greater than the minuend, do the converse, subtract 50 from 55, [and there] remains 5[10]. Thus, in our notation:

$$50 - 55 = 50 + (-55) = -5$$

Bianchini uses the term "additio minuenda" (addition that diminishes) for such operations. This "diminishing" may go as far as to give as result a number smaller than zero. Subsequently, Bianchini will operate with numbers "smaller than 0" just as with the positive ones, respecting the rules of signs.

In the *Arithmetica* two terms appear in relation to negative numbers: "diminutum" and "minus".

The following is an example of the use of "diminutum":

> [...] in the subtrahend there is the root of 24 d i m i n i s h e d, which you have to add to the root of 6[11].

But when Bianchini formulates the "law of signs", he no longer uses the term "diminutum" but the term "minus", that means the n e g a t i v e n u m b e r (considered as an abstract, existing independently from the physical reality, also independently from the physical space, and thus, not meaning, for instance, a "negative direction"). The idea of negative numbers that clearly results from Bianchini's *Arithmetica* (cf. the example of substraction given above, and then, below, the use of the term "minus" in the "law of signs") allows us to suppose that in operations with polynominals with negative coefficients (subtractive numbers) Bianchini was also aware that a subtractive number can function as a negative one, expressed as $a + (-a) = 0$. Furthermore, as it is signalled, Bianchini admits negative products, calling them the "products minus".

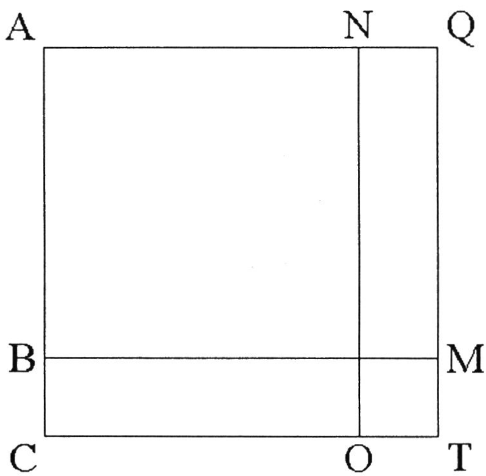
2. EUCLIDEAN *SPATIUM*, BIANCHINI'S CONCEPT OF "NEGATIVE PRODUCT", AND THE GEOMETRICAL "PROOF" OF MULTIPLICATION "PLUS TIMES MINUS"

It is interesing to note that the law of signs appears in the context of multiplication of roots, the multiplication of surds included. Chapter 13 of the *Arithmetica, De practica radicum adinvicem*, in which the law of signs is presented, begins with the definition of multiplication of discrete numbers, then a [geometrical] definistion of multiplication of continuous quantities is given, followed by the "laws of signs"[12].

Bianchini's laws of signs are formulated in the way we use to find in modern text-books of arithmetic. Here, I will quote the second of Bianchini's laws, the first being trivial (multiplication plus times plus), and the third (multiplication minus times minus) being already discussed elswhere, and I will try to present the way Bianchini uses the *Elements* for the sake of constructing his "geometrical proof", justyfying the multiplication "plus times minus"[13].

> [2] When plus is multiplied times minus or minus times plus, the product will be minus. And this results [from the fact] that the more minus increases or plus decreases, the more the product minus [tne negative product] increases.

The statement [2]: "the more minus increases or plus decreases, the more 'the product minus' [the negative product] increases", strange as it might seem, points the character of Bianchini's concept of number. In fact, Bianchini' statement that "the more the minus (negative number) diminishes the more the plus (positive result) increases", and thus, eventually, in [3] the statement "when minus is multiplied times minus the result is plus", concerns operations with numbers considered as abstracts, and not as expressions of physical reality any more. For this reason Bianchini's "numbers" function here independently of the Euclidean concepts of number and of space. Nevertheless, Bianchini refers to the *Elements* in his construction of the "proof" of the [2]. This "proof" is given in chapter 17 of the *Arithmetica*, entitled *De multiplicatione plus per minus*, where Bianchini considers the following example $(4 + \sqrt{9})(8 - \sqrt{16})$. A remark has to be made to this proposal: in the numerical example obviously $\sqrt{16}$ functions as a negative coefficient (and not a negative number):



$AB = QM = 4$	$AC \bullet AN = \square\ ANCO = ?$
$AQ = BM = 8$	$\square\ AQCT = \square\ AQBM + \square\ BMCT = 56$
$BC = MT = \sqrt{9}$	$AB \bullet AQ = \square\ AQBM = 32$
$QN = (\sqrt{16})$ [!]	$BM \bullet BC = \square\ BMCT = 24$
$AC = (4 + \sqrt{9})$	NQ [quae est Rx de 16!] $\bullet\ 4 = \sqrt{256} = 16$
$AQ = 8$	OT [quae est Rx de 16!] $\bullet$ TM [Rx de 9] $= \sqrt{144} = 12$
$AN = (8 - \sqrt{16})$	$\square\ ANCO = \square\ AQCT - \square\ NQOT = 28$

I will multiply 4 plus the root of 9 times 8 minus the root of 16. I draw lines of multiplication at right angle in the point A: AB that is 4 and BC that is the root of 9. Also AQ is 8 and QN is the root of 16.

It is obvious, from that was declared above, that the product AC times AN is an area of a quadrangle ANCO. And first I will look for its quantity. I multiply line AB times AQ and I receive a quadrangle AQBM [...] the area of which is 32. I mul- tiply BM, equal AQ, times BC, which is the root of 9, and this will give the area of the quadrangle. BMCE, equal to the root of 576, i.e. 24; thus, it is obvious that the whole area AQCT is 56, from which the area NQOT has to be subtracted.

So multiply NQ, the root of 16, times QM that is 4 and the product will be root of 256.

Equally, I will multiply OT, the root of 16, times TM the root of 9, and the product will be the root of 144. These two areas taken together are [equal] to the root of 784, i.c. 28, that subtracted from the whole area AQCT, which is equal to 56, the rest is the area ANCO 28[14].

3. EUCLIDEAN *SPATIUM* AND THE POWERS AND ROOTS

Problems with the Euclidean *spatium* manifest themselves at the beginning of chapter 8 of the *Arithmetica*, entitled *De practica in radicibus universalibus operanda*, where an explication of terms is given. Bianchini's discussion of powers and roots, the surd roots included, is conducted on two levels, arithmetical and geometrical. Bianchini uses the terms "finis" and "pronomen" for "power" and the term "radix" for "root". Sometimes however, "radix" means both "root" as well as, what we call a "number as considered in its first power":

Root means the same as the priciple or origin or foundation, and obtains its name from its determined end. Sometimes this determined end [number] is looked for by means of the root that is given, and sometimes on the contrary, the root is looked for by means of the given number. [...]

> A surd root is found by means of l i n e s o r p l a n e s o r s o l i d s, with geometrical demonstrations[15].

In chapter 9 of the *Arithmetica, De tribus generibus radicum* [...], Bianchini considers square roots, cube roots and the "related roots" – "radices relatae", and in this fragment he seems to assert that only these three kinds of roots exist:

> On the square root in numbers or in a r i t h m e t i c. All [each] number multiplied by itself is the root of this product.
>
> In g e o m e t r y each line multiplied by itself constitutes a square that is circumscribed by four equal lines and [has] four right angles. The line e x p r e s s i n g the q u a n t i t y of the s u r f a c e is its root or its square root.
>
> Secondly on the cube root. I state that a r i t h m e t i c the cube root means a number that is multiplied by itself and they multiplied by this product, and thus the cube root of the [ultimate] product is this [first] number.
>
> In g e o m e t r y a line multiplied by itself produces a square surface that multiplied by a s q u a r e surface [!] produces a cube [!].
>
> Thirdly on the related root. I state that a root related absolutely means a root related to its own root. In a r i t h m e t i c it is understood as a number multiplied by itself and the product multiplied by the root of the [first] number [$a \cdot a \cdot \sqrt{a}$]. This number is called the related root of the last product. For instance 4 multiplied by itself gives 16, which multiplied times root of 4 gives the product of 32, of which the related root is 4, and the root of 4, which is 2, I will call the minimum root.
>
> In g e o m e t r y this is understood [as follows]: All square surface times itself constitutes a solid of the four square, equal surfaces. [This solid] multiplied by the r o o t of the first surface forms the u l t i m a t e s o l i d [!], of which the related root is equal to the surface of the first square.
>
> And these solids c a n n o t be well d e m o n s t r a t e d on surfaces (planes), but the examples given in numbers are clear[16].

As it follows from the definitions given above, Bianchini was not always aware that o n l y the square roots and the second powers of numbers do not present difficulties when considered from the point of view of their s p a t i a l character. In the case of a number raised to the third power, the inconsistency between arithmetical expression of the power and a spatial (geometrical) one is to be noted (this inconsistency, however, was noted by Bianchini). Actually, though Bianchini's a^3 expressed arithmetically means

$$a \cdot a \cdot a = a^3,$$

the same a^3 expressed geometrically, as "a line times a line, and times a p l a n e", means

$$a \cdot a \cdot a^2 = a^2 \cdot a^2 = a^4.$$

In fact, Bianchini found himself, while offering a geometrical interpretation of the third power, between the Scylla of the principle of homogeneity (homogenea homogeneis comparari...), and the Charybdis of the lack of a proper g e o m e t r i c a l t o o l to raise a number (expressed as a line segment) to the third power in the framework of the Euclidean geometry.

As for the third sort of roots (and powers), the "related roots", Bianchini admits that in this case the incompatibility of arithmetic with geometry manifests itself plainly. And, after giving this statement, he seems to circumvent this "incompatibility" rather than to face it. According to him, the situation is simply due to the fact that "the demonstration concerning solids cannot be performed on sufaces (planes), but the examples given in numbers are clear"[17]. This statement, closing the discussion, may be interpreted as follows: what matters eventually in arithmetic is the correct solution of an arithmetical problem, given in n u m b e r s . In that way, in the presentation of the radices relatae, the question of the compatibility of the arithmetical expressions with Euclidean *spatium* is simply ignored. Thus, Bianchini discredits here geometry as a discipline "furnishing proofs".

The same tendency to liberate arithmetic from geometry dominates the *Arithmetica algebrae*, the second of Bianchini's mathematical treatises included in the *Flores Almagest*, in which Bianchini considers i.a. operations that lead to the "square of square". When introducing the basic notions of algebra, Bianchini gives the geometrical models that correspond to the first, second and third power. As for the "square of a square", the case is not discussed:

> In the whole practice of the rules of algebra four denominations or four names are commonly used, namely res, census, cubus and census de censu. Res means root, census means a square or a square plane (surface), cubus means a solid. Census de censu is a square of a square. All these originate from a root or from res[18].

Generally, Bianchini, when operating with algebraic expressions, algebraic fractions included, is not disturbed by the lack of geometrical entities that would correspond to them. In there circumstances, he sometimes simply signals the insufficiency of Euclidean *spatium*, when the arithmetical operations lead to the results "overpassing" the dimensions of physical reality – and sometimes he remains in doubt as to the admissibility of such operations rather than as to validity of Euclidean concept of *spatium*. The latter reaction seems to indicate that Bianchini was aware of the serious philosophical consequeces of the apparently trivial arithmetical problems: the admission of powers and roots of a degree higher than the third would require the admission, in the framework of Euclidean concepts of number and of space, of entities corresponding to such mathematical objects. This admission, at its turn, would equally contradict the sensual evidence, and the Euclidean concept of *spatium* that resulted from it. Thus, all declaration in favor of mathematical objects such as a^4 or $\sqrt[6]{a}$ would question the status of Euclidean geometry considered not only as a model of scientific thinking, but also as the model of physical reality. One can easily imagine that Bianchini, being not only a mathematician but also an astronomer, was by no means interested in a similar resolution, since it would also question the validity of his own geometric (kinematic) models explaining the univers[19].

CONCLUSION

As it results from the *Arithmetica* and *Arithmetica algebrae*, Bianchini was aware of the formal insufficiency of Euclidean geometry as related to arithmetic and algebra, both considerably developed by himself. Bianchni's concept of number, extended to negative numbers, and his free use of the powers and roots, surpassed the possibilites of geometrical representations offered by Euclid. In spite of this insufficiency of the *Elements* Bianchini had to accept the concept of *spatium* inherent to the Euclidean geometry. He just limited himself to ignoring the Euclidean *spatium* when it presented a hindrance in justifying the arithmetical operations with "non-Euclidean" numbers. Otherwise, Bianchini remains with Euclid, aware (as it seems) of the fact that the complete liberation of mathematics from the Euclidean *spatium* would signify the liberation of mathematics from reliance on sensual experience: a situation hardly acceptable for a 15th century astronomer working with geometrical models of the univers.

Since 15th century geometry obviously lacked tools to express the concept of number as it is present in both, *Arithmetica* and *Arithmetica algebrae* Bianchini's arithmetic and arithmetized algebra found themselves in a sort of vacuum with regard to geometry (except for the idea of the unit segment that appears in the *Arithmetica*).

Two centuries later, new ways to present relations between number and magnitude, thus the relations between number and space, and consequently between mathematics and physics, are presented in Descartes' *Geometry* (1637). Newton's concepts of space will be created in the framework of these new relations.

Notes

[1] The quotation of Einstein comes from: *The Problem of Space, Ether, and the Field in Physics*, p. 476; where it is situated in the context of Einstein's reflexion on the evolution of the concept of space from "prescientific thought" to the Riemanian spaces, (Ibidem, pp. 473–484). Definitions of the "Euclidean Spaces" see Kiyosi Itô (Ed.): *Encyclopedic Dictionary of Mathematics*. 2 nd Ed. Cambridge Mass. and London 1987. Vol. 1 nr 139 (VI.3) p. 549 and nr 140 (VI.4) p. 554. On the foundations of analytic geometry – and in this context on the concept of n-dimensional Euclidean space E_n, and

n-dimensional Cartesian space C_n, see K. B o r s u k : *Multidimensional Analytic Geometry.* Warsaw 1969 pp. 5–6 and 13–24; The intuition of space, see J. D i e u d o n n é : *History of Algebraic Geometry. An outline of the History and Development of Algebraic Geometry.* Monteray, Calif. 1985 p. 1. On the connectionbetween the intuition of space and a logical contruction in mathematics see H. W e y l : *The Continuum. A. Critical Examination of the Foundation of Analysis.* Dover Publ. New York 1994, p. 49; J. D e m b e k: *Przestrzeń i nieskończono ść. Koncepcja matematyki H. Weyla i jej realizacja w poj ęciu przestrzeni jako kontinuum.* Kraków, OBI 1994 pp. 124-160.

[2] I. G r a t t a n - G u i n n e s s: *Numbers, Magnitudes, Ratios, and Proportions in Euclid's Elements: How Did He Handle Them?* "Historia Mathematica" Vol. 23:1996 pp. 356-365; W.R. K n o r r : *The Evolution of the Euclidean Elements. A Study of the Theory of Incommensurable Magnitudes and Its Significance for Early Greek Geometry.* Reidel, Dordrecht/Boston 1975 pp. 172; K. Z o r m ba l a : *Gauss and the Definition of the Plane Concept.* "Historia Mathematica" Vol. 23: 1996 pp. 420–422 (Chapter 1. *The Concept of the Plane and its Definition in Euclid's Elements*). W. K u y k : *Complementarity in Mathematics. A first Introduction to the Foundation of Mathematics and its History.* Reidel, Dordrecht 1977 p. 87; J. G r a y : *Ideas of Space. Euclidean, Non-Euclidean, and Relativistic.* Clarendon Press Oxford 1979 pp. 5–46, the book is devoted to the "problem of parallels"; B. A. R o s e n f e l d : *A History of Non-Euclidean Geometry. Evolution of the Concept of a Geometric Space.* Springer-Verlag, New York 1988 (first published in 1976, in Russian): Chapter 4 "Geometric Algebra and the Prehistory of Multidimensional Geometry" pp. 152–180. The space in physics was treated by M. J a m m e r : *Concepts of Space. The History of Theories of Space in Physics.* Harper & Brothers, New York 1960 (2nd edition, enlarged and revised). The mathematical space as present in Newtonian physics was subject of discussions concerning Newton's "absolute space". The Cartesian roots of Newton's idea of space as well as its further developments are considered i.a. by R. D u d a : *Newton and the Mathematical Concept of Space.* In: *Isaac Newton's "Philosophiae naturalis principia mathematica"*. W.A. Kamiński (Ed.): World Scientific. Singapore, New Jersey, Hong Kong 1988 pp. 72–83; The classic concepts of the space, considered from a philosophical point of view, are discussed by C. D e i c h - m a n n : *Das problem des Raumes in der griechischen Philosophie bis Aristoteles.* Halle 1893, and by E. F i n k : *Zur ontologischen Fruehgeschichte von Raum-Zeit-Bewegung.* Den Haag 1957.

The subject of this study was inspired by the theme of the 30 Kölner Mediaevistentagung organized by the Thomas Institut of the University of Cologne, namely: Raum und Raumforstellungen im Mittelalter. The one-page summary of my paper, originally entitled *The Space in Mathematics: The Euclidean 'Spatium' as Faced by a Fifteenth-Century Mathematician*, was included in the Symposium materials. When the deadline approached to submit the complete text of my contribution for publication in the Actes of the Symposium (*Miscellanea Mediaevalia*), I was mistaken about the date. It was just before sending the text to Cologne that I realized I was one month late. Therefore I decided to transmit the typescript to the "Kwartalnik Historii Nauki i Techniki" (this final version of the text differs sightly from the one intended for the *Miscellanea*). I apologize to the

Organizers of the 30 Kölner Mediaevistentagung, and I thank the Editorial Board of the "Kwartalnik HNiT" for accepting my paper for publication.

[3] J. Dieudonné: op.cit. p. 4. B.A. Rosenfeld: op.cit. pp. 156–161. Simon Stevin: *Arithmétique*. Leiden 1585 p. 167. On the extension of the number concept in the 15th century mathematics see G. Rosińska: *A Chapter in the History of the Renaissance Mathematics: Negative Numbers and the Formulation of the Law of Signs (Ferrara, Italy ca. 1450)*. "Kwartalnik Historii Nauki i Techniki" Vol. 40:1995 nr 1 pp. 7–8, where Bianchini's geometrical "proof" of multiplication minus by minus is considered. Eadem: *The "fifteenth-century roots" of modern mathematics. The unit segment. Its function in Bianchini's De Arithmetica, Bombelli's Algebra... and Descartes' La Geométrie*. "Kwartalnik Historii Nauki i Techniki" Vol. 41:1996 nr 3–4 p. 64. A study of the development of the concept of negative numbers is given by H. Gericke: *Zur Geschichte der negativen Zahlen*. In: *History of Mathematics, States of the Art. Flores quadrivii – Studies in Honor of Christoph J. Scriba*. Academic Press, San Diego 1996 pp. 279–306.

[4] G. Rosińska: *Algebra w środowisku astronomów krakowskich w XV wieku. Traktat z Flores Almagesti Jana Bianchiniego*. "Kwartalnik Historii Nauki i Techniki" Vol. 39:1994 pp. 3–19 (and the English version forthcoming in the "Organon"), as well as my papers on Giovanni Bianchini refered to in the note 3.

[5] G. Rosińska: *Decimal Positional Fractions. Their use for the Surveying Purposes (Ferrara 1442)*. "Kwartalnik Historii Nauki i Techniki" Vol. 40:1995 pp. 17–32.

[6] See for instance S.A. Jaywardene: *The Influence of Practical Arithmetic on the Algebra of Rafael Bombelli*. "Isis" Vol. 64:1973 pp. 510–523.

[7] See H. Gericke, op.cit. pp. 280–281.

[8] Bianchini's concept of number, expressed through a proportion 1 : = b : ab, together with its application to the multiplication of fractions, as well as its place in the evolution of the number concept during a period from 16th to 18th century: Bombelli – Descartes – Newton, is discussed in G. Rosińska: *The "Fifteenth-Century Roots" of Modern Mathematics*, op.cit. pp. 58–60.

[9] Bianchini: *Arithmetica*. Quando fractiones dividuntur per integra, multiplicentur fractiones secundum proportionem unitatis ad numerum divisorem. [...] In omni multiplicatione tria requiruntur: numerus multiplicans, secundo multiplicandus et productum. Cuius probatio est divisio, quia si productum dividatur per multiplicantem exibit multiplicandus et econverso, si per multiplicandum exibit multiplicans. Et hoc in quantitate discreta.

See also above, note 8. All Bianchini's texts are quoted following the critical edition of the *Arithmetica*, based on the manuscripts preserved in Italy: Bibliotheca Apostolica Vaticana, Vat. Lat. 2288, ff. 16–25v and Vat. Reg. Lat. 1115, ff. 38r–52r; Perugia, Biblioteca Comunale Augusta, 1004, ff. 1r–8r. In Poland: Cracow, Biblioteka Jagiellońska, BJ 558, ff. 1r–12r, and in France: Paris, Bibliotheque Nationale, BN. Lat. 1025, ff. 6r–23r. See also G. Rosińska, *A Chapter*, pp. 5–6 and 14 notes 8–10. The spatia in Bianchini's texts are mine.

[10] Bianchini: *Arithmetica.* [...] Debes ergo de 50 plus subtrahere 55. Et quia numerus subtrahendus excedit numerum a quo debet subtrahi, fac econverso, subtrahe 50 de 55, restant 5. See G. Rosińska, *A Chapter*, pp. 12 and 18 n. 29.

[11] Bianchini: *Arithmetica.* Prout in proposito: in parte subtrahenda est radix de 24 diminuti, quam debes applicare radici de 6. $(-\sqrt{24} + \sqrt{6})$.

[12] The rules of computation with the subtractive numbers (negative coefficients appearing in binominals) go back to Diophantos, 3rd century AD. The first formulation of the rules of computation with negative numbers appears in Europe centuries later, in a 10th century treatise. This treatise, however, according to historians did not exert influence on next generations of mathematicians. M. Folkerts: *Pseudo-Beda: De arithmeticis propositionibus. Eine matematische Schrift aus der Karolingerzeit.* In: "Sudhoffs Archiv" 26 1972 pp. 22–43. H. Gericke, op.cit. pp. 288–290. J. Sesiano: *The Appearance of Negative Solutions in Mediaeval Mathematics.* In: "Archive for History of Exact Sciences" Vol. 32 1985 p. 106.

[13] Bianchini: *Arithmetica.* Cap. 13: Quando plus multiplicatur per minus aut minus per plus productum erit minus et hoc patet quia quanto minus augetur aut plus minuetur tanto productum fiet minus. Multiplication "minus times minus" was discussed in my: *A Chapter*, op.cit. pp. 7–8.

[14] Bianchini: *Arithmetica.* Cap. 17: Rursus volo multiplicare 4 plus radix de 9 per 8 minus radix de 16. Firmabo lineas multiplicationis supra punctum A ad angulum rectum AB, scilicet, quae sit 4 et BC radix de 9. Item AQ sit 8 et QN fuit radix de 16.
Manifestum est per id quod supra declaratum est quod productum AC per AN est superficies quadranguli ANCO, cuius primo quaero quantitatem. Produco enim lineam AB per AQ et fiet quadrangulum AQBM lateribus aequedistantibus et contra se positis aequalibus, cuius superficies est 32.
Item producam BM quae aequatur AQ per BC, quae est radix de 9, et producitur superficies quadranguli BMCE quae est radix de 576, id est 24, quare manifestum est quod tota superficies AQCT est 56, a quibus minuenda est superficies NQOT.
Multiplica ergo NQ, quae est radix de 16, per QM quae est 4, et fiet productum radices de 144. Quae duae superficies simul iunctae sunt radices de 784, id est 28, quare tota superficies NQOT est 28, qui subtracti a tota superficie AQCT quae est, ut supra 56, restat superficies ANCO 28, quod est propositum.
Bianchini: *Arithmetica.* Cap. 17: Cuius regulam accipe per modum supra dictum, videlicet multiplica 4 per 8 erit productum 32. Item 8 per radicem de 9 plus producitur radix de 576 plus. Item 4 per radicem de 16 minus producitur radix de 256 minus. Item plus radix de 9 per minus radix de 16 producitur radix <de> 144 minus. Adde ergo 32 cum radice de 576 quae est 24 erit eorum summa 56, a quibus subtrahe radices de 256 et de 144, quae sunt 28, restant etiam 28, quod est idem propositum.

[15] Bianchini: *Arithmetica.* Radix idem sonat sicut principium vel ortus aut fundamentum et secundum eius determinatam finem acquirit pronomen. Et aliquando per notam radicem datam quaeritur eius determinatus finis et aliquando econverso, per datum pronomen quaeritur radix ex qua oritur. [...] Surda radix [...] invenitur per lineas aut superficies aut corpora cum geometricis demonstrationibus.

[16] B i a n c h i n i : *Arithmetica.* Radix quadrata in numeris seu in arithmetica. Omnis numerus in se ductus vocatur radix producti. [...] In g e o m e t r i a autem omnis linea in se ducta constituit quadratum circumscriptum a quatuor aequis lineis et quatuor rectis angulis, cuius superficiei quantitatis linea, ipsa est radix seu radix quadrata.
Secundo de radice cubica. Dico quod radix cubica in arithmetica dicitur numerus qui in se ductus et iterum in productum, ille numerus dicitur radix cubica istius ultimi producti. [...] In g e o m e t r i a autem linea in se ducta producit superficiem quadratam et etiam multiplicata per s u p e r f i c i e m quadratam producit corpus cubicum.
Tertio de radice relata. Dico quod radix relata absolute intelligitur relata a sua radice, quae in arithmetica intelligitur omnis numerus in se ductus et productus in radice ipsius numeri. Ille numerus vocatur radix relata istius ultimi producti, ut 4 in se ductus producit 16, qui ductus in radicem de 4 producit 32, cuius radix relata est 4, cuius etiam radix quae est 2 nominabo radicem minimam.
In g e o m e t r i a autem intelligitur: Omnis superficies quadrata in se ducta constituit corpus quadratum aequalium superficierum, quod etiam ductum in radice superficiei primi quadrati formatur u l t i m u m c o r p u s , cuius radix relata vocatur superficies primi quadrati.

[17] B i a n c h i n i : *Arithmetica.* Et ista c o r p o r a male in superficie p l a n a possunt demonstrari, sed in n u m e r i s patet exemplum.

[18] B i a n c h i n i : *Algebra.* In tota practica regularum algebrae quatuor denominationes seu quatuor vocabula cummuniter utuntur scilicet res, census, cubus et census de censu. Res enim idem sonat quantum radix. Census autem quadratum sonat seu superficiem quadratam. Cubus vero corpus solidum. Que omnia a radice seu a re oriuntur.

[19] Bianchni's planetary models are essentially Ptolemean, with the exception of the improvement introduced by Bianchini to the Ptolemean model of the Moon.

Grażyna Rosińska

SPATIUM WEDŁUG EUKLIDESA JAKO PROBLEM W MATEMATYCE XV WIEKU

Koncepcja trójwymiarowej, metrycznej „przestrzeni" w *Elementach* Euklidesa była wynikiem zarówno pitagorejskiej koncepcji liczby, jak i odpowiedzi dawanych na trudności, jakie wynikały dla matematyki z takiej właśnie koncepcji. Zatem, Euklidejska przestrzeń (termin „przestrzeń" nie zaistniał jednak w myśli starożytnej, dlatego, respektując szczególność Euklidejskiej „przestrzeni" i w celu uniknięcia wieloznaczności wprowadzono tutaj na jej określenie termin *spatium*), była ograniczona do „przestrzennych relacji" między określonymi p r z e d m i o t a m i . Relacje te były wyrażane poprzez relacje między odcinkami. Innymi słowy, pojęcie „przestrzeni" u Euklidesa było, w pewnym znaczeniu, formalnym wyrazem realnej, trójwymiarowej, zamkniętej przestrzeni, odbieranej w poznaniu zmysłowym [przypisy 1, 2].

Problemy ze *spatium* zaczęły się jeszcze w starożytności i miały swe źródło z jednej strony w formalnych niedoskonałościach Euklidejskiego systemu geometrii (kwestia postulatu „O równoległych") z drugiej zaś strony wynikały z rozwoju arytmetyki i algebry, wcielonych do geometrii (geometria bowiem uzasadniała ich twierdzenia), natomiast w rzeczywistości „nie mieszczących się" w koncepcji geometrii odcinków wyrażających rzeczywistość (w tym przestrzeń) fizyczną, o czym wyżej. Przede wszystkim ta druga sprawa, a także kwestia geometrycznej prezentacji liczb ujemnych, są przedmiotem rozważań w obecnym studium. Jak wiadomo problemy te zostały rozwiązane w XVII wieku w *Geometrii* Descartesa i w dziełach Fermata, dzięki ujęciu relacji „przestrzennych" w sytemie współrzędnych i wyrażeniu przestrzeni „jako całości" (o czym mówi Einstein w zacytowanym fragmencie).

Zanim jednak przyszły te nowożytne rozwiązania, problem stwarzany przez Euklidejskie *spatium* był realną trudnością dla arytmetyki i algebry, operujących już innym pojęciem liczby niż to, któremu odpowiadały *Elementy*. Fakt, że nie było modelu geometrycznego dla potęg i pierwistków wyższych niż trzeci oraz brak koncepcji „odcinka ujemnego", który wyrażałby liczby ujemne (wprowadzone przez Giovanniego Bianchiniego do matematyki już w połowie XV wieku), kwestionował status arytmetyki i algebry jako nauki, bowiem to czego nie można było udowodnić geometrycznie „nie było naukowe". (Tu prawdopodobnie tkwią powody zahamowania matematyki „uniwersyteckiej" w XV i XVI wieku oraz jej rozwój w środowiskach handlowców «scuole d'abbaco», inżynierów i architektów) [przypisy 6, 7, 12].

Sytuacja matematyki w XV wieku w aspekcie jej odniesień do Euklidejskiego *spatium* ukazana jest na przykładzie dwóch traktatów Bianchiniego, poświęconych wykładowi arytmetyki oraz wykładowi algebry. Oba traktaty były już przedmiotem wcześniejszych studiów, mających na celu ukazanie XV-wiecznych źródeł matematyki nowożytnej (wprowadzenie przez Bianchiniego ułamków dziesiętnych oraz liczb ujemnych, traktowanie niewymierności jako liczby, koncepcja „odcinka jednostkowego" i jego funkcjonowanie w wyrażaniu niewymierności) [przypisy 3, 5, 8, 9]. Gdy chodzi o stosunek Bianchiniego do Euklidejskiego *spatium*, to w niektórych przypadkach (jak mnożenie liczb o „różnych znakach" – wg obecnej terminologii), Bianchini wydaje się nieświadomy trudności związanych z istnieniem "ujemnego odcinka", podobnie zresztą, jak przeszło sto lat po Bianchinim, nie był tych trudności świadomy Simon Stevin (w rzeczywistości odcinek w ich dowodach na mnożenie liczb ujemnych jest zawsze odcinkiem dodatnim, konsekwentnie nie ma też mowy o "ujemnej płaszczyźnie"). W innych przypadkach, Bianchini ukazuje nieprawidłowości wynikające z interpretowania geometrycznie wyrażeń arytmetycznych czy algebraicznych i w związku z tym niewystarczalność Euklidejskiej koncepcji *spatium*. Na przykład, gdy mówi otwarcie o niemożliwości przedstawienia geometrycznie działań z potęgami i pierwiastakmi powyżej trzeciego stopnia. Wówczas rolę dowodu spełnia poprawność rozwiązania przedstawionego „w liczbach" [przypisy 10, 11, 13–18].

Wreszcie Bianchini wprowadził szczególną konstrukcję geometryczną z udziałem „odcinka jednostkowego" (ale jej w pełni nie wykorzystał). Konstrukcja ta pojawi się następnie u Bombellego, a u Descartesa stanie się podstawą do zdefiniowania geometrycznie działań arytmetycznych, z uniknięciem trudności „przestrzennych". Będzie to dokonane w ramach geometrii Euklidesa, ale w nowy sposób. Dzięki temu właśnie Descartes stanie się autorem nowego, „całościowego" ujęcia przestrzeni, „niezbędnego dla fizyki Newtona".