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ALGEBRAIC EXPONENTS AND THEIR GEOMETRIC INTERPRETATION *

The history of definitions of algebraic exponents is a most interesting part of the history of algebra and of geometry as well.

Terms like "square" and "cube" for x^2 and x^3 are derived from the Pythagoreans; these called the products of identical factors "square number" (*arithmos tetragonos*) and "cubic number" (*arithmos kybos*). These terms represented special examples of "plane figures" and "solid figures". This terminology was based on figures expressed by geometrical shapes, in which the Pythagoreans regarded units as identical with dots. Later on, the Greeks called square numbers: *dynamis* = *potency*. The Pythagorean terminology is repeated in Euclid's *Elements*.

Hero from Alexandria was the first to introduce in his (*Metrica*) the term x^4, calling it "quadratosquare" (*dynamodynamis*), while Diophantus from Alexandria introduced the fifth and sixth exponent in his *Arithmetics* and called them analogously "quadratocube" (*dynamokybos*) and "cubocube" (*kybokybos*). Diophantus' system of definitions was additive: $x^5 = x^{2+3}$, $x^6 = x^{3+3}$.

We do not know at what time the multiplicative system came into being: "quadratocube" for $x^6 = x^{2 \cdot 3}$. The Byzantine Michael Psellus wrote in a letter, since examined by Paul Tannery, that this system dates back to Anatolius from Alexandria, a contemporary of Diophantus. According to Psellus, Anatolius called x^5 *alogos prôtos* (the first Inexpressible) and x^7 *alogos deuteros* (the second Inexpressible). The Greeks applied the term *alogos*, the Inexpressible to irrational roots which can not be expressed by the ratio of two natural numbers. In this instance, the exponents were not expressible as quotients of 2 and 3. However,

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Psellus may have been mistaken and this terminology may be of a later date. Even so, it is important to know that at Psellus' time this terminology evisted and was known in Byzantium.

We encounter the multiplicative terminology in India. In their *History of Hindu Mathematics* B. Datta and A. N. Singh report that, in *Uttaradhyayana — sutra* (about 300 B. C.) x^2 was called *varga* (square), x^3 *ghana* (cube), x^4 *varga-varga*, x^6 *ghana-varga* and x^{12} *ghana-varga-varga; varga* means "series". This pattern resembles that of Pythagoras. In our opinion, the absence of x^5 and x^7 in this treatise indicates, that the above interpretation is derived from later copyists and that, in reality, the two last-named terms are to mean x^5 and x^7. However, later on the Indians called x^6 *varga-ghana*, and x^5 and x^7 *varga-ghana-ghâta* and *varga--varga-ghana-ghâta*, respectively; here *ghâta* means "product" ($x^5 = x^3 . x^2$). This terminology existed in the 5th century A. D.; in the 7th, Brahmagupta called x^5 *panca-gata* (raised to the fifth), x^6 *sad-gata*, etc.

The Islamic countries employed the additive system exclusively: x^2 is *mâl*, x^3 *ka'b*, x^4 *mâl*, x^5 *ka'b al-mâl*, x^6 *ka'b al-ka'b*, etc. Johannes Tropfke asserts in his *Geschichte der Elementar-Mathematik*, referring to al-Khuwârizmî, that the majority of the Arabic mathematicians chose the Indian system. However, al-Khuwârizmî mentions only the square, although his definition for the free member of the algebraic equation is Indian (*dirham* is the translation of the Indian *rupa*). We find the cube in Banu Musa's writings (9th century). Diophantus' *Arithmetics* were translated into Arabic by Qusta ibn Lûkâ (d. 912), and commented upon by Abûl-Wafâ al-Bûzjânî (940—998); these commentaries have never been found. Al-Bûzjânî's pupil, al-Karajî (al Karkhî), and probably al-Bûzjânî himself in his commentaries, took into consideration exponents higher than the sixth. The infinite series of exponents we encounter in the algebraic treatise of Omar Khayyâm (1048—1131) who in his arithmetical book was the first to extract roots of arbitrary exponents and probably knew the binomial formula (this formula is already given in arithmetic treatises of at-Tûsî (13th century) and al-Kâšî (15th century), while Khayyâm's treatise was never found). The geometric treatise of al-Bûzjânî contains an interesting detail: he constructed a square equalling the sum of three identical squares as a square constructed on the spatial diagonal of the cube built up on the given squares, asserting that an analogous construction is possible even with the number of squares exceeding three. This would mean that, as an example, the side of the square equals the sum of 5 identical squares, built up on top of the spatial diagonal of the quadratocube constructed on top of the given square. In this way al-Bûzjânî considered the quadratosquare, the quadratocube, the cubo-cube, etc. to be multidimensional generalizations of the cube.

However, it seems possible that the Arabic literature contained an Indian terminology also, and that the definitions of x^5 and x^7 were analogous to those of Pseullus. This would explain the word *asamm*, the Arabic equivalent to the Greek *alogos* — meaning "inexpressible"; hence the terms *asamm al-awwal* and *asamm ath-thânî*.

This hypothesis finds some support in the terminology used during the Renaissance by European algebraists.

In Europe we initially also encounter the additive system but, later, only the multiplicative system, for defining exponents. Leonardo Pisano (1190?—1250), a pupil of the Arabs, calls x^2 *census*, x^3 *cubus*, x^4 *census census*, x^6 *census census census*, x^8 *census census census census*. An Italian manuscript from the 15th century writes: x^2 *quadrato* and *censo*, x^3 *cubo*, x^4 *censo di censo*, x^5 *censo di cubo*. On the other hand, in his *Summa de Arithmetica, Luca Pacioli* (1445—1515?) calls x^2 *censo*, x^3 *cubo*, x^4 *censo de censo*, x^5 *primo relato*, x^6 *censo de cubo*, x^7 *secondo relato*, x^8 *censo de censo de censo*, x^9 *cubo de cubo*, x^{10} *censo de primo relato*, x^{11} *tertio relato*, etc., as far as x^{29}. This same terminology was applied in the 16th century by Cardano and Tartaglia: Cardano in his *Ars magna* calls x^5 *relatum primum*, etc. The terms *primo relato* and *secondo relato* are analogous to Psellus' terms *alogos prôtos and alogos deuteros*. The origin of these definitions used by Pacioli is unknown; in our opinion they originated from those used by Psellus: (*ho*) *logos* also means "ratio", a word that can also be translated by *relatum* (the Latin equivalent of the French *rapport*). It is conceivable, that the copyist read *ho logos* because he failed to understand the word *alogos*. In J. Tropfke's opinion *relato* is derived from the Indian *ghâta*; this seems possible on the supposition, that *ghâta* was brought to Europe by the Armenians in whose language the letter *l* is equivalent to *gh* (*Paulos* = *Poghos*, *Solomon* = *Saghomon*, *Lukas* = *Ghukas*, *Baldasar* = *Baghdasar*); the Armenians who in the Byzantian sciences have played an important role, translated the Indian *ghâta* into Latin as *lata*.

The first German Cossists used the additive principle as shown, for example, in the Dresden Manuscript C. 80 (about 1480) where: $x^2 = z$ (*zensus*), $x^3 = c$ (*cubus*), $x^4 = zz$, $x^5 = rzz$, $x^6 = zzz$, $x^7 = czz$, $x^8 = zzzz$, $x^9 = rzzzz$, $x^{10} = zzzzz$. However, already in the Vienna Manuscript Cod. 5277 (about 1500) we find $x^4 = zz$, $x^5 =$ alt, $x^6 = z+c$, $x^7 = c+zz$, where probably "alt" represents an abbreviation of *alogos* (*als* $\rightarrow$ *alt*). In all their later algebraic writings, the Cossists applied the multiplicative principle.

About 1525 a manuscript: "Initius Algebra" was written; yet, in 1524 Adam Riese (Gigas) compiled an excerpt of this manuscript — proof of its being of an earlier date. This Algebra "ist aus Arabischer Sprach in kriechisch transferirt von Archimedo und aus kriechisch in das Latein von Apuleio und verteutsch von Andreas Alexandrus." It seems probable that neither Archimedes nor Apuleius translated this Algebra, but rather that its prototype was an Arabian treatise which reached Germany by way of Byzantium and Italy. In this manuscript we find, for the first time in Europe, the words *affirmativus* (= *positivus*) and *negativus* or *privativus*; these words are probably translations of the Chinese words *cheng* and *fu* and of the words *muthbat* and *manfî* used by al-Qûščî who was a pupil of al-Kâšî and, at the same time, Ulughbeg's ambassador, to China. The last years of his life al-Qûščî spent in Constantinople. In this Algebra and in Riese's work the exponents are called: $x^2 =$ = *Zensus* or *Quadrat* (z), x^3 = *Cubus* (c), x^4 = *Zensus de Zensu* (zz), x^5 = *Sursolidum* (β), x^6 = *Zensicubus* (zc), x^7 — *Bissursolidum* (biβ), $x^8 =$ = *Zensus Zensui de Zensu* (zzz), x^9 = *Cubus de Cubo* (cc); the higher exponents in "Initius" are: x^{10} = zβ, x^{11} = terβ, x^{12} = zzc, x^{13} = quadrβ, x^{14} = zbiβ, x^{15} = cβ, x^{16} = zzzz, x^{17} = quintβ, x^{18} = zcc. The denotations of x^5 and x^7 are explained as follows: "sie nennen *sursolida* d. h. *surda solida*", and Riese asserts: "*sursolidum ist eine taube zal*".

Hence we note that *Sursolidum, Bissursolidum*, etc. are abbreviations of *surdum solidum primum, surdum solidum secundum, etc. Surdum* is the translation of the Arabic *asamm* and the Greek *alogos*; this terminology came into existence from that of Psellus, and from its Arabic analogue. *Solidum* means solid (body); this term is derived from the geometric denominations of the other exponents. Later on we find words like supersolidum used, for example, by J. Peletier (1517—1582). Petrus Ramus (1515—1572) shortened *sursolidum* into *solidus* (x^7 = *bisolidus*, x^{11} = *tersolidus*).

This terminology gave Michael Stifel (1486—1569), in his comment on "the Coss" by Chr. Rudolff (1553), the idea to "über den cubus hinausfaren gleych als weren mehr denn drey dimensiones". Stifel called the Cubus *cörperlicher Punkt*, x^4 = *cörperliche lini*, x^5 = *cörperliche Superficie*, and he considered x^4 to be the result of a moving *cörperlichen Punktes*, etc. This interpretation was applied to the binomial formula also; whed *die Binomia zensica* was split into 4 parts (square of the Prop. II. 4 of Euklid), *die Binomia cubica* in 8 parts ("Kubus von Christoff"), *die Binomia zensizensica* in 16 parts an *die Binomia sursolida* into 32 parts.

Terms like *sursolide* and *B. sursolide* are found in René Descartes' writings, and *sursolids* or *supersolids* in those of John Wallis. This is proof, that the *hypergeometrical terminology* was in use for more than 100 years following the Cossists, and that it has played an important part in promoting the concept of multidimensional space. It should also be mentioned, that Hermann Grassmann's *äussere Produkte* show a close association with Stifel's *cörperliche Linien, Superficies*, etc.