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> Socrates: Well! let us look around because I am afraid not to be understood by some of those averse to philosophy! There are people who think that nothing exists besides that what they can touch and on the level of reality they do not admit neither activities nor generations in few words anything they can not perceive through their senses.
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> Theaetetus: You are quite alright Socrates. These people are rude and difficult to convince.

(1) Watching that class of people quoted by Socrates let us expose the fundamental ideas for a new interpretation of modern mathematics.

In order to put our ideas at their exact position, it is necessary to give, as briefly as possible, a sketch of concepts related with *historical* sciences.

We begin with some generalities. Our ancestors left to us traces of their existence in the world through a marvellous variety of forms: paintings, sculpture, music, writings of all kinds on science, philosophy, poetry, *etc.*; finally, as is usually said, men express ideas through forms. These elements by which we become acquainted with the existence of our forefathers are called historical documents. The study of such documents is the object of various sciences, called auxiliary sciences of History. So, philology studies language in general and its transformations, paleography is related with writing and its evolution, sphragistics is acquainted with stamps, numismatics — with coins, *etc.* All these sciences furnish to historian, when possible, the necessary data he needs. The importance of such information is obvious. They represent to historian what data of experiments do to physicists, with disadvantages of the former in what concerns precision and veracity.

Looking to the mass of historical documents — forms expressing the ideas of mankind — we may ask: are there laws commanding this succession of historical facts, or is absence of organization their own characteristic? This question it the *fundamental problem of philosophy of history.*

The author after various years of researches and meditation is now convinced of the veracity of first assertion, that is, there are laws commanding historical facts.

At this point, to eliminate misunderstandings, we render clear the meaning of the word *law.* By law in History we mean nothing more than a *pattern of reasonings* through which *facts may be arranged.* So, every manifestation of man by expressive forms occupies a very defined place in this pattern, and exactly in this order of ideas we place modern mathematics in a certain position in the set of forms created by western culture.

Different patterns have been proposed by various authors among which we remember Toynbee and Spengler. Toynbee is very well known by his detailed study of several cultures with intention of discovering "cyclic laws" in history and many interesting results were obtained. Spengler became famous by his *Untergang des Abendlands* where also a cyclic theory of history is exposed.

This is not the place for discussing in detail these ideas and we have only to say that our system of history is closely related to that of Spengler. But a significant difference may be pointed out: in his system all expressive forms in a certain culture attain almost together their climax in a certain period — the period of maturity of that culture. Exactly at this point we do not agree with him. We think that, on the contrary, there is a succession of forms in every culture each of them reaching the maximum of expressiveness *in a certain point* of the evolution of that culture and not altogether at the same time. This led us to the *law of evolution of forms* starting from the distinction between *ornament* and *art.*

(2) Let us begin by giving explanations of the words *ornament* and *art* which will be used with different meaning than usual, especially the latter.

A people with a minimum stage of evolution has in general a set of *forms* expressing their common thoughts, their common feelings and their common conception of the world. If those forms constitute a mixed aggregate, in general invariant with time — for example, the same ornamentation remains through centuries — we have a *primitive culture.*

If otherwise, these forms change, following a certain pattern to be defined below, we have a *historical culture* or simply a *culture.*

Now let us analyse with certain details the law of evolution of historical forms in a culture and here the differences between Spengler's system and ours will be clear.

We divide such evolution of a certain form of expression in three stages:

1st stage — called *primitive ornamentation* reflects the feeling of adornment or embellishment, vague memory of primitive rituals for driving away bad spirits and ghosts, and is therefore closely connected with sorcery. The form takes a childish expression, rudimentary, not translating very well the common thoughts; factors related with sensorial perception are preponderant here. Little by little this form grows up, becomes saturated of symbolic content, and enters in:

2nd stage — called art. It becomes a dominant expression, full of life and grandiosity. This is a period of great results and everywhere overflows the immense treasure of its symbolic content.

In general here appear the *great masters* in the field, and the problems and plans of work are very clearly defined. Nobody doubts what to do tomorrow. It seems that forever one will have a constant fountain of pure ideas and forms. The perfect equilibrium is attained. Suddenly, as a reaction to this we reach the romantic movement and so, in few words, the 2nd stage is nothing else but the evolution from the classical ideal to the romantic ideal, and we realize here an exaltation that is nothing else but the expression of a fear coming from the knowledge of that terrible reality: degeneracy and death are approaching us! Therefore every romantic movement is a movement of fear and it is the last great movement of ideas! Finally after a period of great vitality this form begins to decline and enters in:

3rd stage — called *posterior ornamentation*. It becomes then futile and artificious, and more and more intelligible only to a minority and the so called new masters say "that's because it is now intellectual and abstract". It changes rapidly, in the sense that every generation looks to the proceeding one with discredit and more, in the same generation there is no general agreement. The very diagnostic, although lamentable, is that it ceased to be representative and has just given its place to another form, as expression, and, in general, this is not realized at the moment. Finally, it attains a situation of complete absence of symbolic content and in this status it may remain for centuries.

Now we pose the question: is mathematics in western culture a historical form? We are convinced the answer is positive and to see this we will apply our scheme of evolution of historical forms to our culture. After this the meaning of modern mathematics will appear in terms of that law of evolution.

(3) Let us go back to XIth century. Nordic people began to shake off the yoke of archaic roman style, through particular architectonic forms. It is indeed dramatic the fight of styles in Durham Cathedral

(XIth century) or at Caen in Abbaye-aux-Hommes (XIIth century). This revolution is extended until the end of the XIIth century as may be seen, for example, in west face of Chartres Cathedral, when it is born the *Rayonnant* style, observed too in nave and choir.

That is the end of the 1st stage, primitive, ornamentation, in gothic architecture and it enters now at the 2nd stage, art.

It is then marvellous the representative power of the great cathedrals: Bourges, Reims, Amiens, Köln, Saint Chapelle, Chartres, *etc.* The sense of verticalness, of growing up to heaven are there impressed with all vitality and grandiosity of the great symbols. The so called *flamboyant* style is the last manifestation of this magnificent architecture. Indeed, from XVth century on, the gothic style begins to lose its expressive power to give place to another historical form, that some centuries before it had begun to enrich itself of symbolic content—painting.

Architecture enters then at the 3rd stage. It is no more representative but only a sightly *ornament.* Chartres Cathedral *represents* something, but San Giorgio Maggiore Church at Venice is only *well made.* Under this point of view all architecture since the Renaissance to our day is pure ornamentation.

Let us see at this point what happened to painting. When gothic flourished with all its power (2nd stage) painting was limited to frescos and glasses of the great churches, trying a liberation form romanesque style. We see this in the works of Master of San Mathews, where aside golden elements of byzantine art comes up that painfull expression anticipating by centuries "Final Judgement" by Michelangelo. Still in this 1st stage we put Buonaventura Berlinghieri, Magaritone D'Arezzo, Giotto, Cimabue and others.

When in the XVth century gothic enters at the 3rd stage, painting enters at the 2nd stage, that great stage of art. Since Paolo Uccelo and Botticelli until Holbein, Leonardo, Michelangelo, Rafael, Rubens, Velasques and Rembrandt, painting lives with all power and vitality.

Observe that it is no more a *pictorial ornament* of churches, but it has now own life and representative power.

Then came the XVIIIth century. The first troubles came with Watteau and Chardin entering painting at the 3rd stage. We know problems of survival and one of the most serious is: "doing new things never thought before". All tentatives of revitalization fall down and we arrive at the present status of painting, when everybody does what is in his mind and soon protects himself with the shield of theory of art and aesthetics.

Now music comes into action. Since Huchbald (IXth century), Perotinus, Adam de la Halle, Dufay, Orlando de Lassus until Palestrina, music remains at the 1st stage.

With this last and even with Josquin des Pres, in the so-called "golden age of counterpoint", it is very well established what would be music in art stage, but the decisive chords of the 2nd stage appear with Vivaldi, Corelli and so many other geniuses at the end of the XVIIth century and the beginning of the XVIIIth century.

Then appears J. S. Bach, whose work is perhaps the greatest work ever done by only one man. But aside from him we too have Haendel, Mozart, Beethoven and a large succession of geniuses as ever more happened in western culture.

Beethoven died in 1827. Then entered music in the 3rd stage. Since Brahms and Wagner until our time, everybody knows what has happened to music and it is futile to emphasize this point once more.

It must be clear here that we are by no means saying that western culture is formed only by those sequences of forms. Of course a culture is a collection of forms following a pattern of evolution, but we have picked up only one of those sequences, for instance, nothing was said about poetry, theatre, literature and so many other expressive forms — to put in evidence the meaning we intend to give to modern mathematics.

Now we arrive at our main problem: is there some representative form in the 2nd stage today? We think the answer is positive and exactly mathematics is such a form, and starting from this point of view we are now in the position of explaining, in some detail, the meaning of modern mathematics.

(4) Let us go back to the XVIIth century. As we have seen before, music was just beginning to make its entrance. What happened to mathematics? Everybody knows the revolution occurred then with the invention of Infinitesimal Calculus by Leibniz and Newton. As big as this *idea* has been this does not exclude the possibility — as really happened — that the *form* under which it was expressed had characteristics of the form of an *idea* at the 1st stage. Indeed, what meant to Leibniz the Infinitesimals, and — what is more important now — how did he use this concept? If we compare Leibniz's attitude with that of the musicians of the XIIIth and XIVth centuries, we realize the existence of a very striking analogy. Clearly we suppose at this point that the reader is acquainted with the history of music and art in general and also with composition which includes a somewhat complete understanding of harmony, counterpoint and fugue. A comparison with painting or with architecture would lead to essentially the same results at which we intend to arrive here.

If we analyse any reasoning of Leibniz, for example, when he "deduces" the formula relating the differential of the arc with the differentials of co-ordinates, we see clearly the predominance of a strong intuition due more to philosophical considerations than to mathematical ones. With his idea of monads, he naively assimilates the infinitesimal length of arc with the infinitesimal distance of two points.

At the same conclusion we arrive with the analysis of the reasonings of any mathematician from Leibniz until Cauchy. In all these we observe much more care for *calculations* than for *foundations* of mathematics. The following principle is celebrated: "Laissez faire, Laissez passer la foie vous viendra".

The first announcement — still early indeed, of the 2nd stage for mathematics — is given by Gauss' thesis (1799) at Helmstedt University, where he gives the proof of the fundamental theorem of algebra. This is perhaps the first time a mathematician of western culture is more interested in proving a mathematical problem has a solution than in *calculating a solution* of a problem about which it is not known yet if it has *a solution.*

This last attitude is almost dominant, for example, in Euler's *Institutiones calculi integralis* (1768–1770), as well as in all mathematical works until the beginning of the XIXth century. As great as the genius of Euler may have been the form he gave to his mathematical *ideas* which is far from the internal equilibrium of a Mozart's piano concerto or of a Beethoven's quartet.

In a few words, a deeper analysis shows how in that period all characteristics of the 1st stage of evolution of a historical form are fully verified. To render this assertion more striking, we make a comparison with music.

The main preoccupation of musicians from the XIIth century was the polyphonic style where a great predominance of horizontal tendencies is present. Well, the corresponding to *intuition* of mathematics is exactly *melody* in music. A melody is by definition intuitive or it is not a melody at all, because on the contrary it does not say anything immediately to our mind. In the famous treatise of Francon de Cologne translated by Jerome de Moravie (following a reference of Jacques Chailley in *Histoire musicale du Moyen Age*) we read: "Qui vult facere conductum primum cantum invenire debet pulchriorum quam potest".

This is a testimony of the predominance of *melody* over *chords,* the rational and abstract element of music.

At this time all kinds of tricks are used in working a melody of inverted canons, specular canons, crab's canon, stretto, and so on. All this is comparable with the calculations with power series made by Leibniz, Euler, Mac-Laurin and others.

Another feature to call attention here is the preoccupation in maintaining a dynamic character both in music and mathematics. It is sufficient to remember Newton's notation for derivatives in terms of fluxions, directly related to the velocity of a moving point.

In music we have, since the beginning of polyphony, the well known law that forbids parallel fifths for moving voices. In terms of dynamics the explanation for this prohibition is clear: two parallel fifths have no attraction to each other. By another side the dynamic character is attained when a dissonance is resolved onto a consonance. From this we deduce immediately the so-called "principle of resolution of dissonance" that is basic for the study of harmony. It is curious to observe here that explanations ordinarily given in books of harmony for these laws of enchainment of chords are far from being satisfactory. That is because the deep motivations for them have to be searched in history as a general manifestation of a culture. The same holds for books dealing with foundations of mathematics. In no part of these books one can find a comparative study of the important transition from *intuitive* or melodic mathematics to *axiomatic* or harmonic mathematics.

(5) Now let us see what happened to our mathematics since the beginning of the XIXth century.

Abel was one of the first mathematicians in western culture to call attention to the lack of rigour, mathematics possessed — "even an important theorem like Newton's binomial formula is not proved at all for non-integer exponents" — and everybody knows the treatment he gave to power series was quite different from that of Euler.

In general, questions of *existence* of solution rather than *calculations* and questions of *foundations* rather than of blind *developments* became more and more predominant.

At the middle of the XIXth century the work of Cauchy and Weierstrass finally pushed mathematics into the 2nd stage, with the substitution of infinitesimal by limits with the "ε, δ technics"

With Riemann, Dedekind, Cantor we have the formulation of the concept of real numbers in the "*western conception*". Observe here that a theory of such numbers was just given by Eudoxus in the "*old conception*" of geometrical equivalence.

Making again a comparison with music we see the transition from polyphony to *harmony,* that is, horizontal feeling is dominated little by *vertical* feeling.

What is the essence of harmony? One gives a scale, that is a succession of intervals obeying a certain pattern, and begins to form chords by successive adjunction of intervals starting from a certain point of the scale. In classical harmony the "temperated scales" became solidly established and J. S. Bach's "Wohltemperiertes Klavier" is a culminating point. Then we have *laws* telling us how to pass from one chord to another and clearly these laws are the natural evolution of the old principles of counterpoint technics, used in voice movements. In this order of ideas Rameau's *Traité d'harmonie* is fundamental.

The works of old Italian Masters like Carissimi, Corelli, and Vivaldi parallel perfectly to those of Riemann, Weierstrases, Dedekind, Cantor.

With Bach, Haendel, Gluck, Haydn, Mozart and Beethoven we go already to an upper step and their works only now, with modern mathematics, possess a parallel. The complete dominion of axiomatic method in every branch of mathematics: algebra, analysis, topology, geometry, *etc.* is the main characteristic of mathematics in the 2nd stage; it has now a *finality in itself* and all is just fully developed starting from the idea of *set* and then going on through the study of different *structures* defined in it. So, in a certain sense the work of Bourbaki, at least in its conception, parallels the work of Euclid although we do not know if they realize that!

A difficult question, of course, should be that of deciding what are the names of actual living mathematicians who could be put in correspondence with Bach, Beethoven, Mozart or with Leonardo, Michelangelo, Rafael or with any other masters of the *art stage* of some expressive form whatever. The difficulty arrives mainly because we have not yet historical perspective of the actual events and so it should be very easy to draw fanciful conclusions from facts known today.

However, we claim that the romantic movement is the last programme of great style possible for western mathematics. But what is the romantic movement in mathematics? Let us clarify now this important point.

Our intuition of the world is based, among others, on two fundamental ideas: space and time. By the way, the difference of expressive forms in two distinct cultures lies on the diversity of feelings they have about those two fundamental ideas. For instance, that is what distinguishes the Greek world from the Western world, the Egyptian world from the Islamic world, *etc.*

Now the most elementary idea connected with the concepts of space and time is the idea of continuity or of continuous motion. Let us analyse briefly the development of this idea in the western mathematics, according to our principles established above.

From the beginning of classicism (2nd stage of western mathematics) with Cauchy and Weierstrass the idea of continuity is expressed in terms of "ε and δ" and finally after Poincaré, Fréchet, Hausdorff is expressed in terms of neighbourhoods, open sets and all the well known concepts of topology. So all is moulded in terms of sets and the form is precise in the sense that starting from very general ideas we can construct a great form called — by calumny — modern mathematics.

Now if we analyse the definition of continuity, let us say of a function associating to each point of a topological space X a point of a topological space Y, we are involved with the concept of *point,* in the sense that we have a determinist attitude here, namely a function is defined in a *certain point* of X and its value is a certain point of Y. Exactly in this determinist attitude resides the classical ideal in the western mathematics. In few words: the mathematics done in these days is essentially *determinist*, the classical ideal of pure and abstract concepts imposing themselves over nature, over man.

Therefore the romantic movement in mathematics should start with *nondeterminist* assumptions of function, continuity, *etc.*, and this can be done, for instance, by considering not functions from *points* of X to points of Y, but instead, functions from open sets of X to open sets of Y. Then we have to define in a new way continuity, differentiability, etc.

Another interesting question should be that of asking what will be the status of mathematics in two and three centuries from now? Of course, trying to answer this question we are always in danger of arriving at false conclusions, but it is reasonable, if one believes in the first moment that he can use the general law of evolution of an expressive form stated above. By this we should conclude that the next stage for mathematics must be that of *posterior ornamentation*. It is clear that we may stay yet for some time in the art stage, but I am strongly convinced that posterior ornamentation will appear in the near future.

This belief is based on some facts I can see in actual mathematics with some characteristics of a third stage of evolution. That is the case when too much attention is given to very particular techniques, or in few words, when technical problems become an aim in themselves and are not regarded as a means to create new theories. This situation will lead to ridiculous consequences: we shall arrive soon at the creation of a mathematical institute for the study of the sphere of dimension 12,131 where members are so specialized that they do not know a thing about the sphere of dimension 12,132; And if this bad tendency will arrive at the most advanced countries that is because we do not insist conveniently on the philosophical and broad education of the young mathematicians. Since the first years of his university studies he is seduced by the idea of having as soon as possible his Ph. D. and his function as mathematician will be the solution of problems given by someone. There is a real rush for "open problems". That is of course essential: most important ideas in mathematics were born in very particular and specific problems, but these ideas *will never come to the mind which is not philosophically and carefully educated.* Of course, the great artists have also been masters of technique — remember Leonardo, for instance — but they had always a very clear idea of the importance of the technique as a *mean* to attain *something.* It is the same situation of all those tremendous *artifices* of counterpoint from the epoch of Bach whose names are only known by experts and even so, as an historical curiosity.

This remark applies also for mathematicians today and each one is able to verify if it is going to be considered an *artist* or an *artifice* in the ensuing centuries!

After all, in this discussion about he meaning of modern mathematics we clearly do not agree with the exposition of most books dealing with history of mathematics.

What do we find in these books? Nothing more than a catalogue of mathematical events. One starts from the assumption that mathematics "was born" in Egypt or Mesopotamia, then passed through Greeks and Romans and finally became more or less well established in its foundations with "modern mathematics". That modern mathematics is well established, it is clear, as we have seen, but well established in the western sense. Greek mathematics attained also its well established stage in the sense of that culture. Generally, if this is not seen it is because one forgets mathematics is a form of human expression and being so, varies from one culture to another. This is very well discussed in Spengler's work.

It would be very interesting to study mathematics in other cultures following our pattern of evolution of expressive forms. We have done some investigations on Greek mathematics and it seems to hold well. Naturally, we have more difficulty in finding documentation in this case then for our culture. The case of Egypt is more difficult because the few documents we have on its mathematics do not allow us to conclude anything with certainty and so, like a physicist with lack of satisfactory experiments, we cannot say very much.

Needless to say, the ideas exposed here may be applied to aesthetics, theory of art and their relations to mathematics and other manifestations of human's spirituality, and indeed one has to make a great effort not to be attracted definitively by these fascinating studies.

APPENDIX

Here we introduce a table which gives a general view of what has been said in this paper.

For the West we put in the same row the simultaneous events in architecture, painting, music and mathematics and we can observe that by the time one expressive form was in the art period the others were in different periods. So, for instance, if we look to music in art period we see that mathematics was not yet in the art period and so we can say that, as far as precision and perfection of form is concerned, Bach's achievement was much better than Newton's.

We call attention also to the fact that we emphasize for music and mathematics a distinction in the art period between classical and romantic ideals. It is interesting to observe that in our mathematics we still have available a romantic period before we enter in the period of posterior ornamentation, which will be the end of western mathematics as expressive form.

We have put Greek mathematics with corresponding periods in the same row as for western mathematics to make clear the line of evolution of both. We notice that great mathematicians appeared in the romantic period of greek mathematics and so we can expect the same for our mathematics.

We believe that this table will also show the main differences between Spengler's system and ours.

Table

West				Greece
Architecture	Painting	Music	Mathematics	Mathematics
P. O. (IX–XII) Durham, Abbaye-aux-Hommes (Caen).	P. O. Old frescos in churches, *etc.*	P. O. Old church hymns, Huchbald, *etc.*		
Art (XII–XIV) *Rayonnant* to *flamboyant*: Bourges, Reims, Chartres, Köln	P. O. (XII–XIV) Master of San Mathews, Giotto, Cimabue, Berlinghieri, *etc.*	P. O. Old master of counterpoint, church tonalities, Perotinus, Adam de la Halle	P. O. Leonardo Pisano, *etc.* (1202)	
Post. O. (XIV on) San Giorgio, Maggiore, Milan Cath, *etc.*	Art (XIV–XVI) Botticelli, Leonardo, Michelangelo, Rafael, Velazques, Rembrandt, *etc.*	P. O. (XIV–XVI) Orlando de Lassus, Palestrina, Josquin de Pres, golden age of counterpoint	P. O. Scipione del Ferro, Cardano, Tartaglia, negative and imaginary numbers	P. O. (VII–VI B.C.) Thales of Miletus, Anaximander
	Post. O. (XVI on) Watteau, Van Gogh, Modern Painting	Art Classical (XVII–XVIII) Bach, Haendel, Haydn, Mozart Romantic (XIX) Beethoven, Schubert, Brahms, Wagner	P.O. (XVII–XVIII) Cavalieri, Newton, Leibniz, Fermat, Pascal, Descartes, Euler, Lagrange, Infinitesimal Calculus	P.O. (VI–IV B.C.) Pythagoras, Anaxagoras, Theaetetus, Hippocrates of Chios, Hippias of Elis, Archytas
		Post. O. (XX on) Modern music	Transition period (XIX) Gauss, Cauchy, Riemann, Dedekind	Transition period (IV B. C.) Eudoxus of Cnido
			Art (XIX–XX) Classical Cantor, Hilbert, Lebesgue, Bourbaki Romantic ?	Art (IV–II B. C.) Classical Euclid Romantic Archimedes, Apolonius of Perga, Nicomedes
			Post. O. ?	Post. O. (II B. C.––IV A. D.) Diocles, Menelaus, Ptolemy, Pappus

P. O. — Primitive ornamentation; Post. O. — Posterior ornamentation.