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A 2^n-NUMBER SYSTEM IN THE ARITHMETIC OF PREHISTORIC CULTURES

This work has originated from reflections on two particular objects:
—the sexagesimal number system of the Babylonians,
—the arithmetic of non-metric physical units.

1.1. There are separate theories on the formation of the sexagesimal number system of the Babylonians (Wilkosz, Struik, Aaboe). An ancient understanding of the year composed of 360 days, a division of the circle into 6 equal parts by its radius, and the need of a convenient multiple of a Sumerian and an Akkadian measure of weight are alternatively considered to have been involved, and the excellent divisibility of number 60 is supposed to have been decisive for its viability. Yet, along with the sexagesimal system the decimal system was used, and there is firm evidence that practical calculations were carried out in the decimal system, whereas the use of the sexagesimal system was restricted to theoretical considerations of pure mathematics and astronomy (Aaboe). This casts doubts on the theory relating the origin of the sexagesimal system with the practical need for a common unit of weight. Nor is it safe to assume that it was an odd local invention. Notably, a sexagesimal system governed an Old Chinese chronology. "The ancient Chinese reckoned their days, months, and years, by a sexagenary cycle formed by the combination of 10 celestial stems and 12 terrestrial branches. Each double name in the cycle consists of a stem name and a branch name. In order to complete the cycle of 60 the 10 stem names are repeated six times alongside the 12 branch names which are repeated five times." And, "some of these cyclical names were supposed to have originated as far back as the 27th century B.C." (*Encyclopaedia Britannica*). It has also been found that "Sumerian names of numbers are based not on a sexagesimal or duodecimal system, but partly on a quinary ($6 = 5+1$, $7 = 5+2$), partly on a decimal ($30 = 20+10$), partly on a vigesimal system ($40 = 20 \cdot 2$, $50 = 40+10$)." This has led to a suggestion that the Sumerians inherited the sexagesimal system from some other unknown population (*Collier's Encyclopedia*). The enigma of its formation unsolved, one could well ask why a number system of so high a base should have come into use at the very beginning of mathematical thought (Kulczycki). The good divisibility of number 60 is certainly a poor compensation for the inconvenience of a multiplication table running up to 3600, which in practice required the constant use of mathematical tables to perform multiplications and divisions. In fact, the disappearance of number systems others than decimal has been explained by the inconvenience of too long or too short series of basic numerals (Milewski). With the sexagesimal number system a reverse process has to be taken into consideration: it came into use in spite of an essential inconvenience. And this cannot be explained by a natural development of the skill of counting on fingers and toes.

1.2. Physical units were until quite recently not concordant with the decimal number system, and many units used up to now, notably those used by the Anglo-Saxons, are still not concordant. The list of non-metric units extracted from *Mała Encyklopedia Powszechna* shows a certain regularity of their arithmetical construction.

China	Area:	1 mou = 60 ch'ih
	Weight:	1 chin = 2^4 liang
Japan	Length:	1 ri = 36 chô; 1 ken = 6 shaku
Spain	Length:	1 legua = $2^3 \cdot 10^3$ varas, 1 vara = 36 pulgadas
	Area:	1 caballeria = 60 fanegas
	Volume:	1 cahiz = 12 fanegas, 1 fanega = 2^2 quartillas
		1 cantara = 2^3 azumbre
	Weight:	1 arroba = 2^{-2} quintales
France	Length:	1 perche = 18 pieds, 1 toise = 6 pieds
		1 pied = 12 pouces = 144 lignes
Romania	Length:	1 mili = $2^2 \cdot 10^3$ stangene
	Volume:	1 kilo = 2 mirze = 2^3 bannizi
		1 dimerla = 2^{-4} kilé = 2^4 oke, 1 oka = 2^2 litre
Holland	Weight:	1 scheeplast = $2^2 \cdot 10^3$ pound
		1 pound = 2^4 onsen = 2^5 looden
Denmark	Area:	1 tonde = 2^3 skjepper
	Volume:	1 ohm = $2^4 \cdot 10$ pott, 1 kande = 2 pott
Sweden	Area:	1 tunnland = 2^5 kappland
	Length:	1 stang = 2^3 aln = 2^4 fot
Germany	Area:	1 Quadrat-Rute = 144 Quadrat-Fuss
		1 Quadrat-Fuss = 144 Quadrat-Zoll
	Volume:	1 Scheffel = 2^4 Metzen
	Weight:	1 Pfund = 2^4 Unzen = 2^5 Lot
Switzerland	Length:	1 perche = 2^4 pieds, 1 lieue = $2^4 \cdot 10^3$ pieds
	Volume:	1 Saum = 2^2 Eimer, 1 Mass = 2^2 Schoppen

Great Britain and United States		
	Length:	1 statute mile = 2^3 furlongs = 1760 yards
		1 yard = 36 inches, 1 foot = 12 inches
	Area:	1 square mile = $2^6 \cdot 10$ acres
		1 rood = 2^{-2} acre, 1 acre = $2^2 \cdot 10$ rods^2
	Volume:	1 last = $2^6 \cdot 10$ gallons, 1 barrel = 36 gallons
		1 quarter = 2^3 bushels = 2^5 pecks = 2^6 gallons
		1 gallon = 2^2 quarts = 2^3 pints = 2^5 gills
	Weight:	1 pound = 2^4 ounces = 2^8 drams
Poland	Length:	1 mila = 2^3 stai, 1 łokieć = 2 stopy = 24 cale
	Volume:	1 antał warszawski = 18 garny = 2^{-2} beczki
		1 ośmina = 2^{-3} beczki, 1 czasza = 12 garncy
		1 ćwierć = 2^3 garncy = 2^5 kwarty
		1 kwarta = 2^2 kwaterki, 1 korzec = 2^7 litrów
	Weight:	1 korzec = 6 pudów, 1 skrupuł = 24 grany
		1 cetnar krakowski = $2^4 \cdot 10$ funtów

There are pure 2^n sequences as 1 gallon = 2^2 quarts = 2^3 pints = 2^5 gills, decimal multiples of 2^n numbers as 1 square mile = $2^6 \cdot 10$ acres, or units related to the sexagesimal or duodecimal number systems as 1 caballeria = 60 fanegas or 1 foot = 12 inches. Data on weights and measures of ancient cultures show a similar arithmetic (*Encyclopaedia Britannica*), and a prehistoric system of weights belonging to the Culture of the Indus Valley (Mohengo-Daro, Harappa) is reported to have operated the following multiples of a unit equal to 0.8565 gram: 2, 2^2, 2^3, 2^4, 2^5, 2^6, $2^4 \cdot 10$, 200, $2^5 \cdot 10$, $2^6 \cdot 10$, $2^4 \cdot 10^2$, $2^5 \cdot 10^2$, $2^6 \cdot 10^2$, $2^3 \cdot 10^3$, $2^7 \cdot 10^2$ (Volodarski). The result of a historical tendency in the development of physical units is clearly the concordance with the decimal number system. What was the starting point? Must it be taken for granted that a discordance in counting and measuring goes back to the very beginnings of these human activities? There is another fact in the early history of mathematics which could be meaningfully related to the facts here discussed, namely, a binary technique of performing multiplications and divisions by a successive doubling and halving of decimally denoted magnitudes employed by the Old Egyptians. This technique, described as primitive and rudimentary (Aaboe), was effective enough to be commonly used in medieval Europe—the so called "*duplicatio*" and "*mediatio*" method, and even more recently—the Russian peasants method (Wilkosz, *Encyclopaedia Britannica*). That again reminds us that a successful decimal arithmetic—which to become such had to wait for the introduction of the zero sign—is historically a late development as well.

The facts brought here to notice will be explained as diverse manifestations of a single prehistorical reality. The meaning of "prehistorical reality" is here "unrecorded—and speculative." Theories on the past are doomed to be speculative. Theories that have to refer to documents or other information belonging to later periods than that under investigation are likely to be in a higher degree speculative, but do not differ essentially from other theories on the past. And they are common. This theory is further handicapped by a methodological "blackout," as the process involved evades description. It will, however, be judged by its technical merits and, as all theories on the past, by present preferences.

2.1. Consider the octagram on Fig. 1. Let the radii of the octagram denote the sequence of 2^n numbers for $n = 0, 1, 2, \ldots 7$. The combinations of 8 radii in sets of

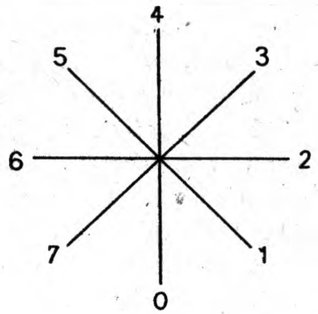


Fig. 1

1, 2, ... 8, taken as sums of 2^n numbers, denote the numbers from 1 to 2^8-1, and the figures of a 256-number system. It may be defined as a binary-contracted-to-2^8 number system, and its apparent binary interpretation is that the direction of radius gives the position of the binary digit 1, whereas the absence of radius—the position of zero.

Example:

$$(162)_{10} = 2^7+2^5+2^1 = (10100010)_2 =$$

The Arabic figures from 0 to 9 written as sparingly as on Fig. 2 give an average of 3.1 graphic elements per sign. The decimal numbers from 1 to 255 contain $9+2\cdot 90 +3\cdot 156 = 657$ figures and $657\cdot 3.1 = 2037$ graphic elements. The respective number of elements for a 2^n number system is $n\cdot 2^{n-1}$, which for $n = 8$ gives $8\cdot 128 = 1024$ graphic elements. The numbers from 1 to 255 are in the English language ex-

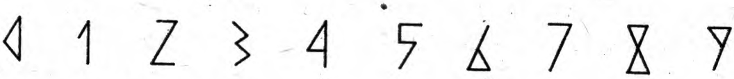


Fig. 2

pressed by 1160 syllables. Accepting for the 2^8-system one syllable per element, the respective total of syllables would be 1024. The 2^8-system is thus considerably shorter in script compared with the decimal system in Arabic notation and would be shorter in speech compared with decimal numeral systems of Indogermanic languages.

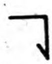
The effort of script and speech can be further lessened if every three and more successive elements are expressed by two elements according to formula

(1) $$2^n+2^{n-1}+\dots+2^{n-k}=2^{n+1}-2^{n-k}$$

Example:

$$2^5+2^4+2^3+2^2+2^1+2^0=2^6-2^0=63$$

Denoting negative values by a broken radius the above example will be given as

63 =  = 

Numbers exceeding 255 may be denoted according to the principles of positional number systems or the system may be developed to a $2^{2\cdot 8}$ clock-dial-modelled system with powers from 8 to 15 denoted by signs as shown in the examples below, and understood as $\pm$ big 0, $\pm$ big 1, … $\pm$ big 7.

$256=2^8=$  $257=2^8+2^0=$  $319=2^8+2^6-2^0=$ 

$1800=2^{11}-2^8+2^3=$  $20217=2^{14}+2^{12}-2^8-2^3+2^0=$  .

A more economical notation based on a 16-radius symbol is possible, but it would require preprinted patterns of the symbol to be filled in with heavy strokes. The system will be denominated the "solar number system," its numbers—the "solar numbers."

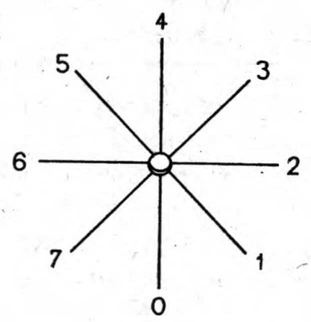
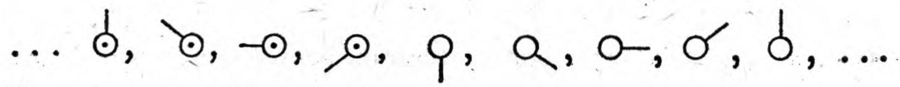


Fig. 3

The solar number system is related to a number system represented by the octagram on Fig. 3. Let the radii of the octagram denote the sequence of 2^{2^n} numbers for $n = 0, 1, 2, \dots 7$. The combinations of 8 radii in sets of 1, 2, ... 8, taken as products of 2^{2^n} numbers, denote all 2^n numbers from 2^1 to 2^{255}. The system will be denominated the "natural number system," its numbers—the "*N* numbers." A solar number is thus the sum of *N* numbers which are products of 2^{2^n} numbers.

The sequence of *N* numbers ... $2^{-4}, 2^{-3}, 2^{-2}, 2^{-1}, 2^0, 2^1, 2^2, 2^3, 2^4, \dots$ may be now denoted as



or

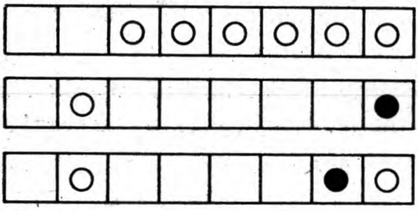
It is well to note that number $2^{255} = 5.8 \cdot 10^{76}$ is inconceivably large (larger, in fact, than the estimated number of all atoms in the universe) and that *N* numbers could be denoted analogically to solar numbers up to 2^{65535}, but most of them would never become useful.

2.2. With numbers constructed in this way, the arithmetic invented for the decimal system disappears, a game emerges instead. A draught (checker)-board is needed and two sets of draughtsmen, say, white and black. A set of symbols to transcribe the game may be useful as well—if the course of the game is to be communicated to those not in a position to look on (as it is in chess-columns of newspapers). For the sake of conveniency the bottom row of squares to play on will be described at first—and with a slight modification in names. "Draughtsman" is much to long. "Man" would be better, but might have adverse implications ("white man"—"black man"). So let us call the pieces in game just "*N*". The rules of the game are simple:

1. A double *N* on any square may be replaced by a single *N* on the next left hand square.

2. Two opposite *N* on any square cancel each other.

To make the game useful (or mathematical) it is only to set for the *N* on the right hand corner square "one." It is now apparent that integers from 1 to 255 in both colours can be played on the bottom row of squares of a draught-board. Any multitude of *N* (shortly: *MN*) gives thus an integer, if there is a game (there may be the case of an empty board, which is obviously not a game, zero being thus excluded), but any integer may be played by some *MN*. For instance, the integer 63 may be represented as



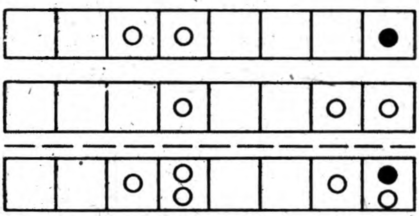
or

or

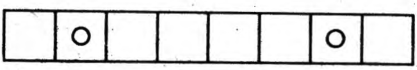
etc.

Because of this variety of representation players agreed upon playing *MN* in a convenient form called "number." The meaning of "convenience" as here used might be described as "not diminishing the pleasure of play," and it could be associated with the concept of "least effort," since considerable effort in play diminishes pleasure indeed.

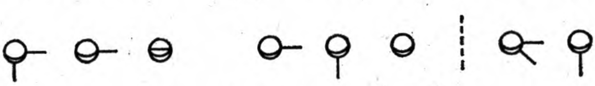
Now, it is obvious that *MN* may join and disjoin. If there are two or more *MN* to be joined, we play "junction" by simply putting them together



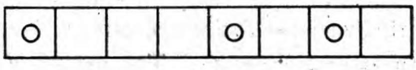
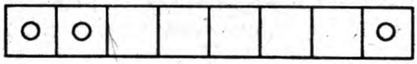
and transforming the resulting *MN* according to the rules of the game into a convenient *MN* called "number."



This number is the integer 66, and the play has its analogue in the arithmetical operation of addition 47+19 = 66. This, however, would be a very bad notation if used to transcribe the game. Instead, we will use the solar notation

 which clearly indicates that for the first *MN* there are *N* on the sixth and fifth square, and an opposite *N* ($\overline{N}$) on the first square, and similarly so for the other *MN*. We may as well use the symbols of the natural number system  which give the place of the square by the solar configuration of strokes, with the reservation that the first square (bearing "one") is not counted.

It is also seen in play that disjunction of an *MN* into two or more *MN* may be played by picking off some *MN* from the initial one, or by joining opposite *MN* ($\overline{MN}$), since "picking off" is just "cancelling of opposites." It would as well be easy to show that variants of the game analogical to other arithmetical operations could be played—and in different ways.

If, for instance, the *MN* to join is  and the "number" of junctions  the game may be set on the board as shown by the *N* given in dotted line on Fig. 4. To play the game it is only to move the *N* by a chess-bishop's move on to the bottom and left hand edge of the board, and reduce the resulting *MN* to the "number" shown by the *N* in full line. With "dense" configurations of *N* it is, however, convenient to carry out the reduction before reaching the edge of the board.

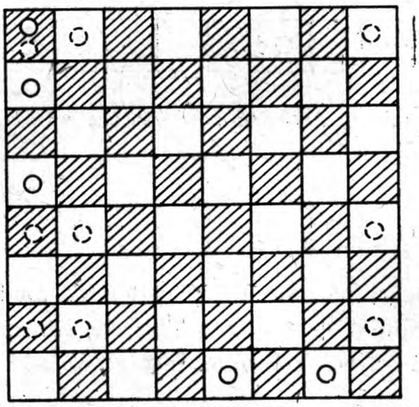
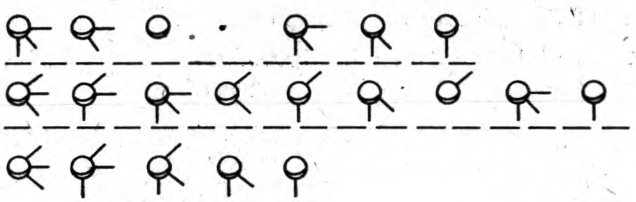


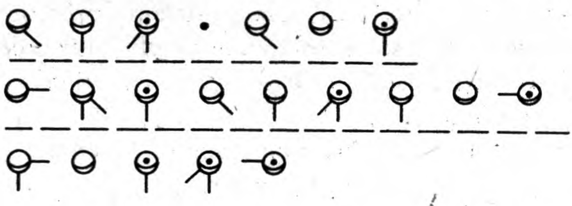
Fig. 4

The arithmetical analogue of this game is the operation $193 \cdot 138 = 26634$. Incidentally, it may be noticed that by building a "castle" of three *N* in the upper left hand corner-square of the 8^2-square board products up to $2^{16}-1$ or 65535 can be obtained. The solar transcription of the game is 

but the natural transcription gives a better insight into the game



Indeed, this transcription fits so well to the game that it might be played merely in symbols! This will be demonstrated by playing an *MN* of junctions of another *MN*, with both *MN* containing fractional *N* (obtained by a series of halvings of "one").



The arithmetical analogue of this play is the operation $6.125 \cdot 2.5 = 33.6875$.

A closer look on the display of symbols will bring us to notice that this is a game, too, similar to the game of cards. With sequences of eight cards in the pack, numbers from $\pm 2^{-8}$ to $\pm 2^{8}-1$ could be played, with sequences of thirteen cards—numbers from $\pm 2^{-13}$ to $\pm 2^{13}-1$. And it would not be necessary to put the symbol on the card, since any convention on sequence and opposition would do perfectly well.

This exercise of constructing a number system and its "arithmetic" shows that for certain sufficient large *n* a 2^n- number system can be notated economically while maintaining the unique operational advantages of a 2^1 system and its suitability for computation on devices. If some such number system would be put at the beginning of a development which ended with the universal acceptance of the decimal-number system, the sexagesimal and duodecimal-number systems would perfectly fit into the process.

3.1. To transform decimal numbers into solar numbers the following simple procedure could be applied. (The binary representation used below refers directly to both solar numbers and configurations of *N* on an n^2-square board.)

Decimal number of the second order of magnitude

$73 = 7(2^3+2^1)+3$

$$\begin{array}{r} 111000 \\ +\quad 1110 \\ +\quad 11 \\ \hline 1001001 = \end{array}$$

Decimal number of the third order of magnitude

$$491 = 4\ (2^6+2^5+2^2)+9\ (2^3+2^1)+1 \qquad \begin{array}{r} 100000000 \\ +\ 10000000 \\ +\ 10000 \\ +\ 1001000 \\ +\ 10010 \\ +\ 1 \\ \hline 111101011 = \end{array}$$

For decimal numbers of the third order of magnitude the procedure becomes laborious and it will become much more so with numbers of higher orders of magnitude. A further deficiency is the infeasibility of an inverse operation to transform solar numbers into decimal numbers.

Now, let us consider the following arrangement of N numbers

$$\begin{array}{cccc} 128 & 32 & 8 & 2 \\ 64 & 16 & 4 & 1 \end{array}$$

Every two successive numbers of horizontal alignment give the sum $2^{n+2}+2^n = 5 \cdot 2^n$. The following sequence of multiples of ten is obtained: $2^0 \cdot 10$, $2^1 \cdot 10$, $2^2 \cdot 10$, $2^3 \cdot 10$, $2^4 \cdot 10, \ldots$ A number of the decimal system is transformed into a number of the solar system by decomposing it into a sum of N multiples of 10 and a rest < 10, and by performnig some further operations without any effort of memory.

Example:

$$491 = 49 \cdot 10 = (2^5+2^4+2^0) \cdot 10+1 \qquad \begin{array}{r} 101000000 \\ +\ 10100000 \\ +\ 1010 \\ +\ 1 \\ \hline 111101011 = \end{array}$$

On an n^2-square board doubles of N are set for $2^n \cdot 10$ and are put into position $2^n(2^3+2^1)$ with chess-knight's moves.

An inverse operation to transform numbers of the solar system into numbers of the decimal system is now possible; the result, however, may be obtained quicker by memorizing the position of elements (radii, N) giving sums equal to numbers resulting from following relations:

	Numbers:
(2) $2^{n+2}+2^n = 5 \cdot 2^n$	5, 10, 20, 40, 80, ...
(3) $2^{n+4}-2^n = 3 \cdot 5 \cdot 2^n$	15, 30, 60, 120, 240, ...
(4) $2^{n+3}+2^{n+2}+2^{n+1}+2^n = 3 \cdot 5 \cdot 2^n$	as above

(5) $2^{n+1}+2^n = 3\cdot 2^n$ 3, 6, 12, 24, 48, ...
(6) $2^{n+2}-2^n = 3\cdot 2^n$ as above
(7) $2^{n+3}+2^n = 3^2\cdot 2^n$ 9, 18, 36, 72, 144, ...

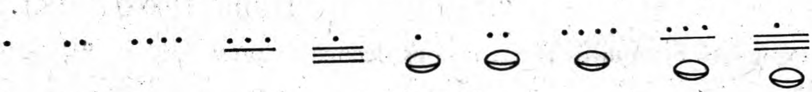
Example:

 $= 480+10+1$ according to (4) and (2).

With some practice and numbers not exceeding several hundreds relations (2) to (7) may be conveniently applied to decimal-to-solar transformations as well.

Equations (3) to (7) give for n ≥ 2 multiples of number 12. The duodecimal--number system preserved in the concepts of dozen and gross as well as in weights and measures explains itself now as a mediating system linking the solar system with the decimal system. A peculiarity of this system is now comprehensible: it never functioned independently, but in association with the decimal system (*Collier's Encyclopedia*).

3.2. Consider the sequence of numbers: $2^0, 2^1, 2^2, 2^3, 2^4, 2^0\cdot 20, 2^1\cdot 20, 2^2\cdot 20, 2^3\cdot 20, 2^4\cdot 20, \ldots$ In the Mayan vigesimal notation these numbers appear as



The system is better organized and more effective in transformations of its numbers into solar numbers compared with the decimal system, since transformations of numbers from 1 to 20 are easily made by heart, and numbers to be decomposed into a sum of N multiples of 20 are only half as large.

Example:

 $= 491 = 24\cdot 20+11 = (2^4+2^3)\cdot 20+11$

$$\begin{array}{r} 101000000 \\ +\ 10100000 \\ +\ \quad 1011 \\ \hline 111101011 \end{array} =$$ 

With this advantage of a broader base, vigesimal numbers twice as large as decimal numbers can be easily transformed to solar numbers. A memorized decomposition of numbers larger than 2^7 would, however, become tiresome which limits the range of direct solar $\rightleftarrows$ vigesimal transformations to about 2000. With number systems of still broader bases this range could be further extended.

3.3. Consider the following arrangement of N numbers

$$\begin{matrix} 128 & 64 & 32 & 16 \\ 8 & 4 & 2 & 1 \end{matrix}$$

The sequence of $(16-1)2^n$ numbers provides the bases for easy-to-solar-transformable number systems. They enable a memorized solar $\rightleftarrows$ $(16-1)2^n$ transformation of reasonable large numbers of the first order of magnitude and a direct transformation of numbers of the second order of magnitude. Relations (2) to (7) match perfectly with those bases and can be most effectively used. A transformation of numbers of those systems into decimal numbers is relatively simple. And there is the excellent divisibility of the $(16-1)2^n$ bases for $n \geqslant 2$.

Example of a $(16-1)2^3 \rightleftarrows$ solar transformation:

$$(58.112)_{120} = 58 \cdot 120 + 112 = 58(2^7-2^3)+112 = 7072 \qquad \begin{array}{lr} & 1110100000000 \\ - & 111010000 \\ + & 1110000 \\ \hline & 1101110100000 = \end{array}$$

The inverse operation is simply

$$\begin{array}{lr} & 110111'0100000 \\ + & 11'0111000 \\ + & 11000 \\ \hline & 111010'1110000 = (58.112)_{120} \end{array}$$

but its explanation requires the laborious decimal expression $2^{12}+2^{11}+2^9+2^8+$ $+2^7+2^5 = 2^{12}-2^8+2^8+2^{11}-2^7+2^7+2^9-2^5+2^5+2^8-2^4+2^4+2^7-2^3+2^3+$ $+2^5 = 2^5(2^7-2^3)+2^4(2^7-2^3)+2^3(2^7-2^3)+2(2^7-2^3)+2^6+2^5+2^4 = (2^5+2^4+$ $+2^3+2^1)(2^7-2^3)+2^6+2^5+2^4 = 58 \cdot 120+112$. It is now easy to notice that broader $10 \cdot 2^n$ bases such as 40 or 80 would lack some of those advantages. Moreover, they are psychologically adverse because of their increasing deviation from the actual N numbers, and less convenient in transformations of solar numbers (the "inverse operation") because of the proximity of 2^3 and 2^1.

A possible vestige of the virtual base of those number systems is a special term for number fifteen found in some languages (for instance, the German *Mandel*).

A $(16-1)2^2$-system is equivalent to the sexagesimal system of the Babylonians. This system has been operated as decimal-sexal and duodecimal-quinary system (preserved in our division of time) which would confirm the use of schemes (2) to (7) in memorized transformations of numbers up to 60. The decimal-sexal notation of the Babylonians facilitated at the same time the transformation of sexagesimal numbers into decimal numbers. The directly transformable numbers of the second order of magnitude reaching up to 3599 were sufficiently large to solve most of the practical problems, and the positional relativeness of the Babylonian notation could indicate that this range of the system was used similarly to a slide rule technique.

The limit of memorized transformations extended up to $(16-1)\ 2^3$, a number system with number 120 as base is obtained. Vestiges of a duodecimal-decimal system preserved in the ancient concept of "hundred," later "long hundred" equal to 120 (*Collier's Encyclopedia*) might suggest that such a number system was actually used. Number 120 is to be understood as the base of a number systém giving every number up to $(2^7-2^3)^2-1$ as $n_2 \cdot 12 \cdot 10 + n_1$, with n_1, n_2 transformed by heart ($n_1, n_2 < 120$). On a n^2-square board solar $\rightleftarrows$ $(16-1)2^3$ transformations are carried out mechanically, since it suffices to play 128 as 120+8, or 120 as 128−8.

3.4. The transition from the 2^n-number system to the decimal system hardly discernible in mathematics is petrified in physical units. Described in mathematical categories it is little more than a speculation, here it is still a reality. And this difference in "actuality" is comprehensible. Physical units are extremely stable and conservative—the present troubles of the Anglo-Saxons remind us of that convincingly enough, but much more so must it have been in a world of restricted communication. The metric system consolidated historically late. "The essentials of the system were embodied in a report made to the French National Assembly by the French Academy of Science in 1791. The metre, the unit of length was to be the one ten-millionth part of the meridional quadrant of the earth ... It took many years for the metric system to be adapted as obligatory in France. Its progress in most other countries has also been slow." (*Encyclopaedia Britannica*), and the predominance of the decimal number system is here not so clear as in mathematics. Units concordant with the 2^n number system and later transition number systems are still successfully in use or remain in the living memory of peoples.[1]

A theory should explain the initial facts which have led to its creation and embrace a number of other facts into a consistent entity. A lucky theory reveals sometimes an unexpected meaning in facts and discovers new fields of investigation. A few theories have ruined our understanding of the world.

4.1. The practicable solar $\rightleftarrows$ decimal transition systems (including the $(4+1)2^n$-systems) are the following:

(8) $(4+1)2^n = 5 \cdot 2^n$

(9) $(16-1)2^n = 3 \cdot 5 \cdot 2^n$

(10) $(64+1)2^n = 5 \cdot 13 \cdot 2^n$

(11) $(256-1)2^n = 3 \cdot 5 \cdot 17 \cdot 2^n$

Systems (8) and (9) have been discussed. Systems (11) giving the lowest multiple of 10 as high as 510 seem already inconvenient. Systems (10) have been used. A 260-day

[1] However, one could ask: "For how long?" A table of non-metric physical units in the *Encyklopedia* cited earlier has not been reprinted in a later edition. The board of editors have probably found it obsolete. This reminds us of the trivial fact that a civilization in progress not only produces information, but also loses information with the passing of time. The "to be or not to be" of a theory as here presented can well be a question of a few years' time.

Sacred Year used by the Maya of Central America in chronological and astronomical calculations (Thompson) corresponds with the base of a transition-number system we describe as $(64+1)2^2$.

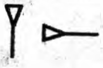
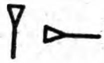
This coincidence could bear the evidence of a 2^n-number system in prehistoric cultures. It could arrive from a discovery of 2^n-denoted numbers or from a discovery of physical units concordant with the $(64+1)2^n$-number systems, or, indirectly, from a better understanding of the chronology of the Maya. Unfortunately, a "better understanding" requires here a substantial reconsideration of the object under investigation. The problem therefore will be discussed in the final paragraph of this work. There are, however, facts marginally described in science or merely noticed which explained in terms of this theory gain in significance while not requiring the painful effort of reconsideration.

4.2. Some numerals constructed according to a non-decimal scheme, such as "eleven," "twelve" are regarded as vestiges of the ancient duodecimal number system. There may well be deeper causes of the irregularity as in the French language the sequence of such numerals goes up to *seize*—2^4. Irregular are also the French numerals for 70, 80, 90, regarded as vestiges of a vigesimal number system (preserved in some other languages as well—the English "score"). However, Celtic, Germanic and Greek have also a break between 60 and 70 (*Collier's Encyclopedia*) and the crucial point could be 2^6.

The numeral "nine" is in many languages near to or homonymous with "new" as in *novem—novus* (Latin), *neuf—neuf* (French), *neun—neu* (German). It could be a remainder of an ancient understanding of "nine"—the new basic numeral in contradistinction to the solar number system which used only eight basic elements.

Another vestige of the solar number system might be the dual number occurring in certain linguistic families. A remark of the linguist J. Vendryes may here be quoted (Vendryes): "The use of the dual number must be regarded as reflecting needs different from those which could result from our mental habits. Today we see no reason to set duality against plurality."

4.3. Many of the games we have inherited are essentially mathematical and can be used to carry out *N*-calculations. Such are the board games draughts (checkers) and halma, such are, as well, dice, domino, and playing cards. "Checkers was played in the days of the earlier Pharaohs" (*Encyclopaedia Britannica*) and there is an Old Egyptian legend telling of some such game played by the god Toth to win the last 5 days of the year for the goddes Nu (Milewska, Zonn). As for cards, it is known that "early in the T'ang dynasty the Chinese had paper money which Chinese cards so resembled in design that their respective times of emergence could hardly have been long separated. Originally the cards and money may have been identical or, since in many societies gaming implements preceded money, either may with equal likelyhood have engendered the other" (*Encyclopaedia Britannica*). One might have thought that in those ancient times it was hard enough for man to survive. Was it a play-time? There is perhaps a hint in the legend connected with the creation of chess telling of a reward of wheat grains doubled on each square of the board.

4.4. The Babylonians used three signs to denote their numbers: 1  , 10  , 100  . The signs  and  are identical with the correspondent solar numbers. The sign  could be understood as $1 \cdot 60 + 4 \cdot 10$, with 4  identical with the solar number.

One of the Greek notations of numbers (the so called Herodian notation) denoted 5 by Γ, identical with the solar number.

4.5. The greatest peculiarity and enigma of Egyptian mathematics is its theory of fractions (Wilkosz). The Egyptians represented any fraction as a sum of unit fractions $1/n$ and performed arithmetical operations on fractions only after having brought them to that form. To facilitate the reckoning with such fractions (carried out mainly by doubling) tables were used giving fractions of the type $2/n$ (for odd n up to 101) as a sum of unit fractions (*Historia matematyki*) like $2/5 = 1/3 + 1/15$; $2/7 = 1/4 + 1/28$. The principle by which fractions are decomposed to unit fractions is obscure. It is known that a fraction could not be given as a trivial sum of equal unit fractions, and that among several possible representations always one and the some was chosen. Representations like $2/5 = 1/5 + 1/5$ or $2/5 = 1/4 + 1/12 + 1/15$ are thus excluded. It might seem that the idea of this method was to extract the main part of a fraction. Now, "division," though played on a board easy enough in comparison with our "arithmetical operation," is not as easy as "multiplication," N-fractions (analogical to decimal fractions) are played in multiplication just as integers, and fractions of the m/n type being only proposals of a play can not be played at all. To play "division" it would be thus convenient to use tables of reciprocals which, incidentally, give short periods easily learned by heart, as explained by the binary expressions on the right (the stroke over "1" denotes subtractive units):

$1/3 = 1/4 + \ldots = 0.01010101 \ldots$

$1/5 = 1/4 - \ldots = 0.010\bar{1}010\bar{1} \ldots$

$1/7 = 1/8 + \ldots = 0.001001001001 \ldots$

$1/9 = 1/8 - \ldots = 0.00100\bar{1}00100\bar{1} \ldots$

$1/11 = 1/16 + 1/32 - \ldots = 0.00011000\overline{11}00011000\overline{11} \ldots$

$1/13 = 1/16 + 1/64 + \ldots = 0.000101000101 \ldots$

$$1/15 = 1/16 + \ldots = 0.00010001 \ldots$$
$$1/17 = 1/16 - \ldots = 0.0001000\bar{1} \ldots$$

etc.

As long as multiplications with such solar reciprocals are played on a board the manipulations and the estimate of accuracy are easy. With the written "doubling" technique an initial diminishing of the number of terms in the sum of reciprocals would become a necessity. The Egyptian method seems to take advantage of a possibility which might have revealed itself with the use of the decimal system, namely, that any m/n fraction can be expressed exactly by a finite sum of different unit fractions. Instead of $2/7 = 1/4 + 1/32 + 1/256 + 1/2048 + \ldots$ there would be $2/7 = 1/4 + 1/28$. Thus, in case of an m/n fraction with N denominator the Egyptian method would give an N representation. Such case has been quoted in Wussing. For $21/8 = 2\ 5/8$, the Egyptian algorithm is given as

1 —	8	
/2 —	16	
$/\dot{2}$ —	4	The point over 2, 4, and 8 is the hieratic sign denoting unit
$/\dot{4}$ —	2	fractions.
$/\dot{8}$ —	1	The result: $2+\dot{2}+\dot{8}$

The conclusion is that as the decimal system prevailed, N-fractions were expressed through reciprocals of decimal numbers, but within a still binary arithmetic. In categories of the decimal arithmetic the Egyptian method is extremely laborious—and absurd.

4.6. A binary interpretation of a solar number was given earlier. Another interpretation is possible: a solar number is a sum of 2^n numbers denoted by individual signs. It is only to replace "2" by "b" representing the base of any number system, and the Old Egyptian, Old Chinese, Roman, and one of the Greek notations of numbers will comply with this definition. And then there is the binary character of Old Egyptian arithmetic, the incapability of using multi-unit figures found in the Egyptian, Babylonian, and Mayan notations of numbers, and the use of subtractive figures in Roman and Babylonian numbers. Curiously, these particulars added to our picture of ancient arithmetics as primitive science encumbered with magical meanings (magic number seven, magic square).

4.7. "Zero" not used in N-mathematics is historically an astoundingly late invention. The Babylonians, who developed a positional number system, marked "zero" by an empty place between signs. This weakness of arithmetic contrasts strongly with the overall mathematical skill of the Babylonians which in some fields was surpassed only after three thousand years (Wilkosz, Bourbaki). One might say there was a reluctance to denote zero. Was it so because the concept of a "zero magnitude" was definitely alien to N-mathematics? There is another reason to ask this question: the concept of a "zero magnitude" has created quite a specific mathematics—and quite a specific picture of the physical world, but it would require some thought to become aware of that.

One could object to all that has been said here, that with a number of facts available it should not be difficult to pick out some of them fitting in to a theory. There is, however, an "invitingness" to facts and coherency in presentation which encourages one to a wider exploration of its possibilities. Technically it is certainly advantageous, since the facts the present paper embraces have been hitherto explained by separate theories or presented as odd occurrences not meriting the trouble of thorough investigation. Would it prove its validity in confrontation with a scientifically established system? A breakdown of the theory would mean little, a breakdown of the system—very much. It should therefore be defended with all might.

5.1. The chronology of the Maya is based on two calendars:

"a) The 260-day Sacred Calendar (Aztec *tonalpohualli*, Maya *tzolkin*) was for ritual purposes only, and had nothing to do with astronomical phenomena. It was based on the numbers 1 to 13 and the 20 named days, each of which had its own title and glyph. Every possible combination of one day with one number gives a total of 260 pairings before the cycle starts all over again.

b) The Solar Year Calendar of 365 days, divided into 18 months each of 20 days plus a period of 5 'unlucky days.'

Any given day can be expressed in terms of both these cycles, and it will be 52 years (i.e. 73 Sacred cycles or 52 Solar ones) before the two calendars are in phase again and the same combination is repeated. The 52-year period is called the Calendar Round." (Bray, Trump)

Even such encyclopaedic information, which incidentally does not mention another chronological cycle: the 360-day tun governing the so called Long Count, gives an idea of the problems this chronological system unique by its redundancy has brought into being. One could, for instance, ask why the Maya should have chosen a Sacred Year of precisely 260 days, and as this period determines the cycle of 52 Solar Years one might as well inquire about its chronological significance. According to the state of science the answer would refer to magic. Number 260 equals to 20 times 13, and 13 is magical. Why magical? Well, there is the explanation that it perhaps hints at an archaic lunar year of 13 months.[2] This might seem dubious, but it shall

[2] Compare with Schlenther, pp. 89-90: "Die sich immer wieder folgenden Zwanzig ergaben aber noch keine grössere fest zu fixierende Enheit. Aus diesem Grunde wurden in dem mesoamerikanischen Kulturbereich die Zahlen 1-13 hinzugeführt ... In der 13 haben wir wahrscheinlich eine alte kultische Zahl zu sehen, vielleicht symbolisiert sie die dreizehn Monate eines alten Mondjahres ... So kam es aber auch, dass man die verschiedenen Zeiten des Jahres mit magischen Kräften in Verbindung brachte. In erster Linie waren es der Bedeutung nach Regen- und Feuchtigkeitsdämonen und die Winddämonen ... In den zauberisch-animistischen Zeiteinteilungen galten besondere Zeiteinheiten als gut oder schlecht, und zwar nicht nur allgemein, sondern sie waren noch speziell unterteilt in Bezug auf Aussaat, Feldarbeiten, Ernte usw. ... Die Zeitabschnitte liegen innerhalb der 260d des magisch-rituellen Kalenders. Er hat sich, so primitiv er war, bis heute erhalten. Er diente und dient ausschliesslich als Wahrsagekalender."

not be discussed seriously since the question itself by insisting on reason where there obviously is none, seems somewhat insane. Magic, as it is, stands firmly for the "unknown," and as much is unknown in the chronology of the Maya a theory dispensing with magic is likely to meet with a rebuke. It may, however, convince that the chronology of the Maya can be coherently discussed in terms other than magic.

Plate 19 of the Codex Dresdensis (Thompson) gives multiples of 5, 10, 15, 20, and further intervals of 5 up to 60 of number 584, followed by 65, 130, 195, and 260 multiples of the same number, remarkably close to the synodical revolution of the planet Venus averaged at 583.92 days. The cycle of 584 days falls in phase with the 365-day Solar Year each 8 such years, and with the 260-day *tzolkin* every second Calendar Round which gives two vital relationships of the Maya chronology

(12) $8 \cdot 365 = 5 \cdot 584$

(13) $2 \cdot 52 \cdot 365 = 2 \cdot 73 \cdot 260 = 65 \cdot 584 = 37960$

Number 260 has been deduced as the base of a solar $\rightleftarrows$ decimal transition number system. Such transition system would fit as well to the vigesimal system used by the Maya. Both numbers $260 = 2^8 + 2^2$ and $584 = 2^9 + 2^6 + 2^3$ indicate at a common octal order as there is $2^8 : 2^2 = (2^3)^2$ and $2^9 : 2^6 = 2^6 : 2^3 = 2^3$. In fact, the relations (12) and (13) are expressed in the octal number system as

(14) $10 \cdot 555 = 5 \cdot 1110$

(15) $2 \cdot 64 \cdot 555 = 2 \cdot 111 \cdot 404 = 101 \cdot 1110 = 112110$

or

(15′) $2 \cdot 64 \cdot 555 = 111 \cdot 1010 = 101 \cdot 1110 = 112110$

40000 (16384)	4000 (2048)	400 (256)
20000 (8192)	2000 (1024)	200 (128)
10000 (4096)	1000 (512)	100 (64)
4000 (2048)	400 (256)	40 (32)
2000 (1024)	200 (128)	20 (16)
1000 (512)	100 (64)	10 (8)
400 (256)	40 (32)	4
200 (128)	20 (16)	2
100 (64)	10 (8)	1

Fig. 5

Similarly to earlier demonstrations it might be possible to play the chronology of the Maya in a game resembling checkers on an octally arrayed 2^n board as shown on Fig. 5. The values of the squares are denoted both octally and decimally, the former are given in parentheses. Draughtsmen are set for days, and the only rule of the

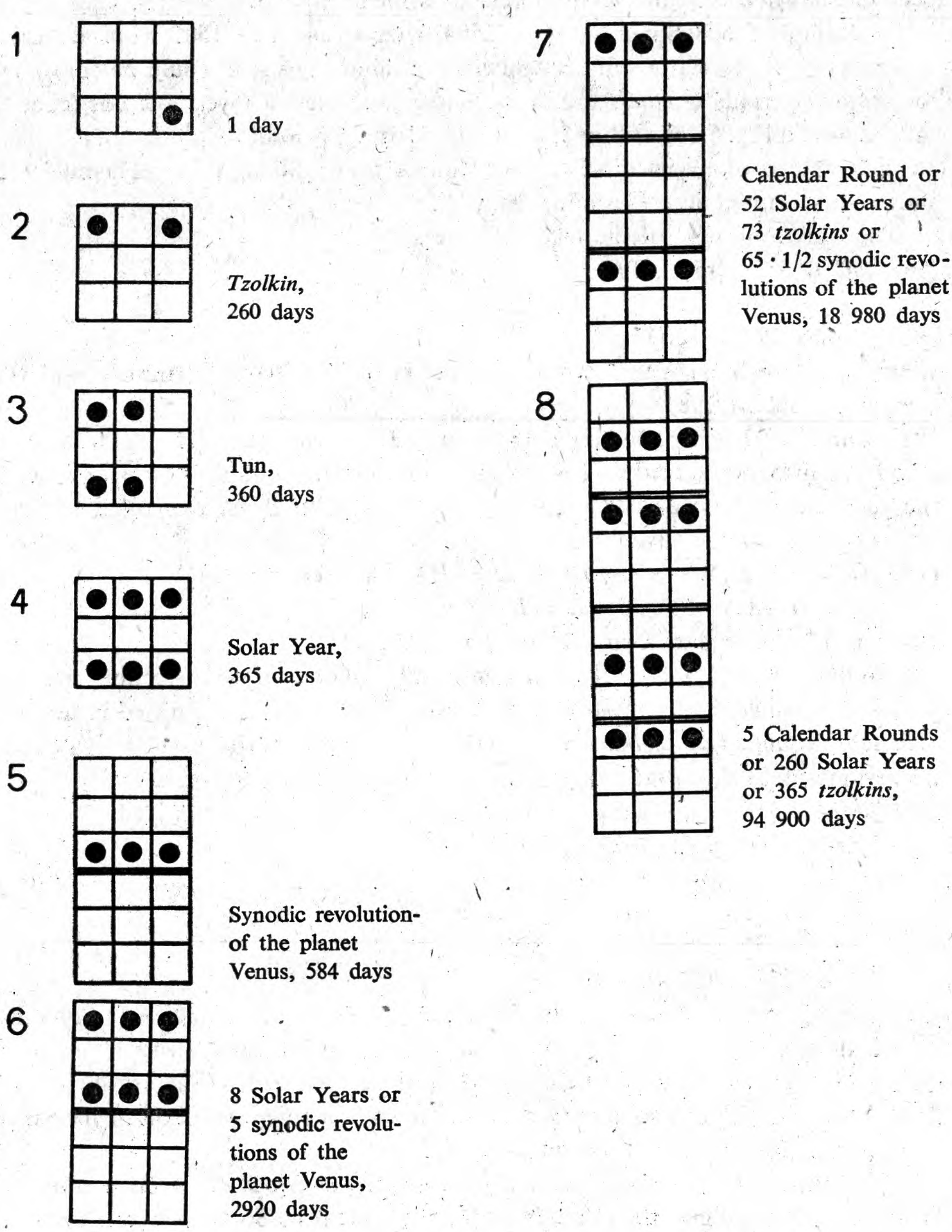


Fig. 6

game is that two draughtsmen on one square can be substituted by one draughtsman on the next "higher" square according to $2 \cdot 2^n = 2^{n+1}$. Chronological periods used by the Maya give the specific configurations of draughtsmen shown on Fig. 6.

According to Spinden time corrections calculated by the Maya "seem to have been one day in four years for short periods while for long periods they made 29 Calendar Rounds (1508 Solar Years or 550420 days) equal to 1507 tropical years." The specific configurations of draughtsmen in mind suggest it would be convenient to compute periods of time, and in particular multiples of 4-year periods (configurations 6–8 on Fig. 6), by multiples of the single triad "synodic revolution of the planet Venus" instead of by multiples of the double triad "Solar Year." Formula (15′) shows this arithmetical feasibility as well.

The Maya estimate of the true year gives the expressions

(16) $1507\ Z_1 = 1508\ H$

and

(16′) $Z_1 = H + H/1507$

wherein H—the Solar Year, Z_1—a true year of 365.242203 days (the tropical year is actually 365.242198 days).

Number 1507 is divisible only by 11 and 137 which would give mathematically inconvenient periods for time corrections. If, however, we diminish the accuracy of the year to $Z_2 = 365.2420$ days or to $Z_3 = 365.242308$ days, two other formulas next to (16′) can be given

(17) $Z_2 = H + H/1508 = H + H/29 \cdot 52 = H + K/4 - 12K/29 \cdot 52$

(18) $Z_3 = H + K/4 - 12K/30 \cdot 52 = H + K/4 - 2K/C$

wherein H—the Solar Year, K—day, C—260.

It follows from (17) and (18) that remarkably accurate time corrections are conveniently calculated with multiples of 52 years. These periods expressed in terms of Calendar Rounds (R), *tzolkins* (C), "weeks" of 13 days (F) (the *tzolkin* divides into 20 periods of 13 days), and days (K) would be according to formula (18)

(19) $52\ Z_3 = R + F - 2/5\ K$

$260\ Z_3 = 5R + 5F - 2K$

.

.

.

$1040\ Z_3 = 20R + C - 8K$

With the less exact formula (17) the third term in (19) would be 8.28 K for 1040 Z_2.

All these periods as well as the 4-year periods within the Calendar Round by which time corrections would be undertaken are represented on the octally arrayed 2^n board by particular configurations of the triad "synodic revolution of the planet Venus" (configurations 6 to 8 on Fig. 6).

The Maya did not use a bissextile year. Glyphs inscribed on stelae qualified as "Secondary Series" give the number of days by which the Solar Year was ahead of the true year. There are also the so called "Supplementary Series" interpreted in terms of lunar months of 29 and 30 days. The same numbers appear in the denominators of formulae (17) and (18). This might invite reconsiderations, but with the scarce texts preserved the case does not seem very hopeful. More likely is the tracing of glyphs pictured within a 9-place-value square denoting numbers from 1 to 2^9-1.

Little doubt, on the other hand, is left on the chronological meaning of the *tzolkin*. It determines the right periods by which time corrections should be undertaken (visualized by the vertical alignment in configurations 7, 8), and gives at once the periods of time by which the count based on the Solar Year should be corrected. As any given day is expressed in terms of both the Tzolkin Calendar and the Solar Year Calendar, this correction is readily reckoned within the former and the result can be immediately given in the latter.

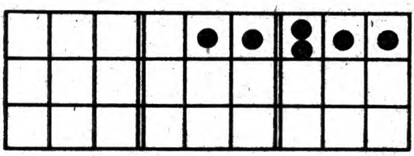
This property of the *tzolkin* chronology, which at first might be perplexing, results directly from its affinity to the 2^n number system. The *tzolkin* differs by a 2^{-6} part from the solar number $2^8 = 256$, and the periods given in (19) are exactly a 2^{-6} part longer than periods equal to 2^7 years and to multiples of 2^7 years which would be used for time corrections within a solarly reckoned chronology to give a true year of 365.24187 days. If such periods were played on the octally arrayed 2^n board, configuration 8 by losing a 2^{-6} part would be represented by the six upper draughtsmen only. This affinity is in fact such that any number of days expressed in terms of Calendar Rounds, *tzolkins*, "weeks", and days displayed on the octally arrayed 2^n board is directly legible as a solar number or an octal number of days. This can

$4 \cdot 10^8$ (2^{26})	$4 \cdot 10^7$ (2^{23})	$4 \cdot 10^6$ (2^{20})	$4 \cdot 10^5$ (2^{17})	$4 \cdot 10^4$ (2^{14})	$4 \cdot 10^3$ (2^{11})	400 (256)	40 (32)	4
$2 \cdot 10^8$ (2^{25})	$2 \cdot 10^7$ (2^{22})	$2 \cdot 10^6$ (2^{19})	$2 \cdot 10^5$ (2^{16})	$2 \cdot 10^4$ (2^{13})	$2 \cdot 10^3$ (2^{10})	200 (128)	20 (16)	2
10^8 (2^{24})	10^7 (2^{21})	10^6 (2^{18})	10^5 (2^{15})	10^4 (2^{12})	10^3 (2^9)	100 (64)	10 (8)	1

Fig. 7

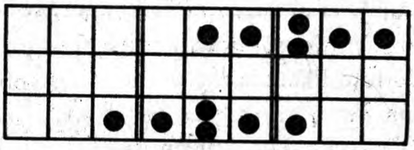
be shown by arraying the 2^n board as on Fig. 7. The numbers are again denoted octally, with decimal values given in parentheses.

The Calendar Round equal to 18 980 days is now given by the following configuration of draughtsmen

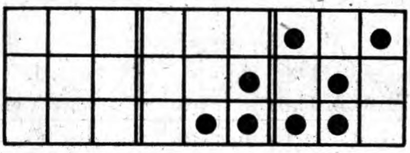


Calendar Round

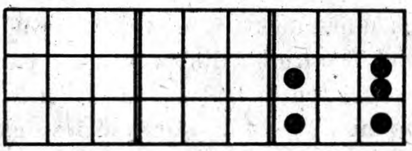
Let us consider, for instance, a period of 17 Calendar Rounds, 23 *tzolkins*, 15 "weeks", and 2 days. On the 2^n board given on Fig. 7 it would be set as follows



1+16 = 17 Calendar Rounds

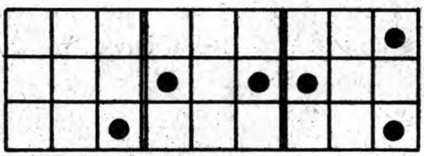


1+2+4+16 = 23 *tzolkins*



5+10 = 15 "weeks", and 2 days

By assembling and reducing according to $2 \cdot 2^n = 2^{n+1}$ the resulting configuration gives the number



$(1\,010\,000\,010\,010\,000\,101)_2 =$
$= (1202205)_8 = (328837)_{10}$ days

If it were now for this given period to calculate by how many days the calendar year of 365 days has run ahead of the true year, there would be according to (19) 17 "weeks" minus $3 \cdot 2 + 4/5$ days for 17 Calendar Rounds. As for the remaining 23 *tzolkins*, a relationship given in the Codex Peresianus (Thompson) could be used

(20) $$5\,H' = 7\,C$$

wherein H'—the period of 364 days, C—*tzolkin*. It gives about 15 H' for 23 *tzolkins* which would require about 4 days to be added. The result is 16 "weeks" and 10 days, or 218 days.

For a 2^n-organized chronology this correction would be given by an expression as simple as

(21) $$T = H_s \cdot 2^{-2} - H_s \cdot 2^{-7}$$

wherein T—time correction in days, H_2—the solar number of calendrical years of 365 days. It will be noticed that the arithmetical operation involves a subtraction only since the divisions are analogical to the shifting of a decimal point. With a 2^n-

organized chronology the *tzolkin* has obviously become meaningless. Not so the triad "synodical revolution of the planet Venus." If it were to calculate the time correction for a given number of days K, the first term in equation (21) could be given conveniently as

$$H_s \cdot 2^{-2} = K/V \cdot 2/5 \text{ [days]} \tag{22}$$

which could be played on the octally ordered 2^n board due to the particular alignment of $V = 584 = (1110)_8$, or more generally, of number 73 and its 2^n multiples. A similar alignment in the other direction gives surprisingly the period of 7 days used in our chronology, but a period of 28 days occurs in the chronology of the Maya as well (Thompson).

With the decimal system the arithmetical operations of (21) would become more laborious (in fact, a simpler but rather crude rule has been chosen to obtain our so called Gregorian year of 365.2425 days), and it would be much more so with the vigesimal system of the Maya, who had to struggle with a multiplication table amounting to 400.

The conclusion is that the *tzolkin* made a 2^n-organized chronology workable within the vigesimal number system.

Concepts of magic, rite, cult, etc. have not been used in this work. Some mathematical concepts of the theory here presented have been introduced instead. They added coherency to a chronological system displaying distinctive marks of perfection in remainders hardly sufficient to permit a reconstruction. By admitting the possibility of a lost perfection there is a risk of introducing logic where there was none. Yet, in my opinion, risk should be taken if the alternative is the adding of disorder. For a good reason. It is extremely difficult to make sense of absurdities, but easy to let order into chaos.

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