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On Axiomatization of Non-Cartesian Logics[1]

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Abstract:
The paper analyses a non-Cartesian logic Sm_4 as a typical sample of rather a wide class of logics which have unseparable pairs of truth-values in their minimal matrices. The algorithm for construction of bivalent semantics, described by Caleiro et al., cannot be directly applied to these logics.

Keywords:
non-Cartesian logic, logical matrix, unseparable truth-values, axiomatization

Nowadays it is generally accepted as a kind of truism by a significant part of logicians that Belnap's four-valued logic B_4 (the definition see below) is a good logical system, which is both useful in practice and fruitful in theory. Lots of papers and monographs deal with the syntactic analogue of B_4, which is a well-known system of First Degree Entailment (FDE), and with its algebraic correlate, id est the class of De Morgan algebras.

A quite disappointing conclusion that one may derive from results of this huge research in the mentioned field is that B_4 is something very much like a piece of hard concrete, not allowing any constructive modification without changing its nature to the extent which cannot satisfy any final user, neither a philosopher, nor a logician.

This observation seems to be plausible in view of the fact that after all manipulations with B_4 the structure of logic itself remains unchanged, if we treat B_4 as the power set of the set of classical truth-values True and False and proceed with taking power sets of our resulting sets, we can, after all, only obtain the same system, characteristic for FDE (see [5]).

In this paper the author considers possible ways, which, as it seems to him, may be interesting in their philosophical and technical implications, of modification for B_4 and other logical matrices, characteristic for FDE. The basic fact we use is the concept of logical consequence in B_4 disguises the existence of distinguished values in matrices, characteristic for this logic. Once we start trying to deal with B_4 as the logic which really has designated values, we can obtain some new logics just with changing the set of designated values of the original logic and, maybe, slightly modifying definitions of logical connectives in matrices, characteristic for FDE, or adding new connectives to them.

At first, we need the well-known definition of Belnap's connectives with following truth-tables:

&	t	b	n	f
t	t	b	n	f
b	b	b	f	f
n	n	f	n	f
f	f	f	f	f

~	x
f	t
b	b
n	n
t	f

Using these definitions of conjunction and negation, we can also define disjunction in the usual way, just putting $A \vee B =_{def} \sim (\sim A \& \sim B)$. The definition of logical consequence is not canonical, because it does not use the notion of designated values and preserving them from premises to conclusion:

$\Gamma \models A$, if and only if for every interpretation of formulae γ in Γ the value of γ is less or equal to the associated value of A with respect to the partial order on the set of truth-values: $f \leq b \leq t$ and $f \leq n \leq t$.

Shramko and Zaytsev, however, in [4] proved that using this definition of logical consequence is equivalent to using a canonical one, putting the set of designated values as {t, b}. But what happens, if we do not want the definitions to be equivalent, if we do not think that changing always means spoiling? In this case one may consider a new logic with Belnap's connectives and a single designated value {t}. With present definitions this leads to reconstruction of all paradoxes of classical logical consequence. Still, with some modifications we can obtain a logic, which is a kind of brand-new.

Until now we only considered formulae, which do not involve an implication-style connective. There is a possibility of adding the implication of Smiley to the set of Belnap's connectives. Smiley's implication $A \rightarrow B$ gives the value t, if and only if the value associated to A is less or equal to the value associated to B; it gives the value f in all other cases. Thus defined connective has the following truth-table (for more information on Smiley's implication see [2]):

→	t	b	n	f
t	t	f	f	f
b	t	t	f	f
n	t	f	t	f
f	t	t	t	t

In addition, we change the definition of negation on the values b and n, now ~b=n and ~n=b. Thus, the new truth-table is:

~	x
f	t
n	b
b	n
t	f

The resulting matrix for this new logic is the following:

Sm_4 = <{t,b,n,f},{t},{~,&,→}> , where {t,b,n,f} is the set of truth-values; {t} is a set of designated values; {~,&,→} is the set of modified logical connectives.

We define logical consequence in the usual way in terms of preserving "truth" from premises to conclusion:

$\Gamma \models A$, if and only if A has the value t, whenever all γ in Γ have the value t.

In contrast with B_4, this new logic Sm_4 is neither relevant, nor paraconsistent any more. Still, its matrix among most of the logical systems described in literature enjoys rather a rare property, which the author of this paper in [3] has called "being non-Cartesian". Non-Cartesian logic is a logic which has at least one pair of unseparable truth-values in its least by cardinality characteristic matrix. The notion of separability for truth-values is used by Caleiro et al. in [1] in their algorithm of constructing bivalent semantics for many-valued logics. Most logics have enough linguistic expressive power to make every pair of truth-values in their minimal characteristic matrix separable. Such logics (id est the predominant part of all finitely-valued logics) can be called Cartesian. The definition of a separable pair of truth-values v_1 and v_2 is the following:

Truth-values v_1 and v_2 are called separable, if and only if

- v_1 is in the set of designated values, if and only if v_2 is not in this set; or
- it is possible to find a formula in the language of the logical system in question such, that this formula only contains a single propositional variable p_i and logical connectives, and the truth-value, assigned to this formula under the interpretation of p_i with one of the values v_1 or v_2, is in the set of designated truth-values, if and only if the truth-value, associated to this formula under the interpretation of p_i with the other truth-value from the pair v_1 and v_2, is not in the set of designated values.

So, if a logic has this separability property for every pair v_1 and v_2 of truth-values in its minimal characteristic matrix, it can be called Cartesian, otherwise it is non-Cartesian. One can easily check that the formulated logic Sm_4 is non-Cartesian, as the truth-values b and n in its matrix, which indeed is a minimal one, cannot be separated using any formula, constructed just with a single propositional atom and any composition of the connectives from the set {~,&,→}. This logic validates all of the axioms and rules of the relevant system E (of entailment), but fails to validate the specific axiom of system R. Therefore, it can be dealt with as an explosive extension of E.

Every logic which has a "Cartesian" minimal characteristic matrix can always be endowed with a "non-Cartesian" characteristic matrix, which cardinality is not minimal, but in such cases it is, obviously, possible and rather easy to get rid of the excessive truth-values. In case of truly non-Cartesian logics, one cannot just throw away any of the elements of non-separable pairs without changing the logic itself. On the other hand, it is possible to add some operators to the language of a non-Cartesian logic to make it Cartesian. In particular, it is enough (if possible) to add all functions $J_i(x)$, where i is an element of the set of truth-values, and $J_i(x) = 1$, if $i = x$; otherwise $J_i(x) = 0$.

What really makes non-Cartesian logics interesting from philosophic point of view is that these logics do not allow direct use of algorithm, formulated by Caleiro and others in [1], for construction of bivalent semantics. This algorithm may be seen as an attempt of constructive realization of Suszko's Thesis, but due to pure existence of non-Cartesian logics one can immediately conclude that this algorithm is far from being universal. This, in its turn, may be viewed as a support to the hypothesis that Suszko's reduction cannot be universally constructive in principle.

Sm_4, however, allows a standard Hilbert-style axiomatization, which consists of axioms and rules of the system E (of entailment) plus a single axiom and two deductive rules:

A+: $A \& {\sim}A \rightarrow B$;

R+1: ${\sim}A / A \rightarrow B$;

R+2: $B / A \rightarrow B$.

Such an axiomatization is semantically adequate for Sm_4, this can be proven using standard methods.

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Note

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