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BUBBLING AND BISTABILITY IN THE IMMIGRATION AND INTEGRATION MODEL

SUMMARY

This paper proposes an analysis of creation mechanism of bubbling sequences and bistability regions in bifurcation scenario of a special class of one dimensional two parameter map. Answer is also proposed to the question, which one of those behaviors is more typical in economic systems.

The above-mentioned considerations will refer to the economic model of immigration and integration.

Keywords: bistability, bifurcation, model of immigration and integration.

Introduction

The studies related to the onset of chaos in one-dimensional discrete systems modelled by the non-linear maps have been quite intense and exhaustive during the last two decades[1].

This paper proposes an analysis of creation mechanism of bubbling sequences (and bistability regions) in bifurcation scenario of a special class of one-dimensional, two parameter map. The answer is also proposed to the question, which one of those behaviours is more typical in economic systems. The simplest cases where those are present are maps, with at least two control pa-

[1] S.N. Elaydi, *Discrete Chaos*, Chapman & Hall/CRC 2000.

rameters, one that controls the non-linearity and the other which is a constant additive one, i.e. maps of the type;

$$X_{n+1} = f\left(X_n, a, b\right).$$

Analyses of the character of observed bifurcations in dynamic systems are more frequent in the literature now.[2]

The bubbling scenario is seen in the bifurcation diagrams of many non-linear systems like coupled driven oscillators,[3] oscillatory chemical reactions, diode circuits, lasers,[4] insect populations[5] and traffic flow systems etc. Bistability is equally interesting and bounded with many non-linear systems like a ring laser and a variety of electronic circuits. A recent renewal of interest in such systems arises from the fact that they form the ideal candidates for studies related to stochastic resonance phenomena. A question arises, what types of bifurcations are typical for this kind of existing economic systems.

In paper "Bubble bifurcation in economic model"[6] was presented the econometric model of costs and production. Analysis of dynamics of this model showed that it generated bubble bifurcations.

The above-mentioned considerations will refer to the economic model with endogenous rates of immigration and integration.

[2] G. Ambika, N.V.Sujatha, *Bubling and Bistability in Two Parameter Discrete Systems*, "PRAMANA – Journal of Physics" 2000, Vol. 54, No. 5, pp. 751–761; H.E. Nusse, J.A. Yorke, *Analysis of a Procedure for Finding Numerical Trajectories Close to Chaotic Saddle Hyperbolic Sets*, "Ergodic Theory and Dynamical Systems" 1991, No. 11, pp. 189–208; H.E. Nusse, J.A. Yorke, *A Procedure for Finding Numerical Trajectories on Chaotic Saddles*, "Physica D" 1989, No. 36, pp. 137–156; A. Tufaile, J.C. Soortorelli, *Chaotic Behavior in Bubble Formation Dynamics*, "Physica A" 2000, No. 307, pp. 336–346.

[3] J. Kozłowski, U. Parlitz, W. Lauterborn, *Bifurcation Analysis of Two Coupled Periodically Driven Duffing Oscillators*, "Physical Review E" 1995, No. 51 (3), pp. 1861–1867.

[4] D.D. Bruns, D.W. Depaoli, C.R. Menako, S. Rajput, *Chaotic Dynamics of Bubble Formation From Electrified Capillaries*, Prepared for presentation at AIChE Annual meeting 2002, November 6 th, 2002 (unpublished); S. Lain, M.F. Goz, *Numerical Instabilities In Bubble Tracing In Two – Phase Flow Simulations*, "International Journal of Bifurcation and Chaos" 2001, Vol. 11, No. 4, pp. 2727–2733; K. Otsuka, J.-L. Chern, *Dynamical Spatial Pattern Memory in Globally Coupled Lasers*, "Physical Review A" 1992, No. 45, pp. 8288–8291; M.C. Ruzicka, *On Bubbles Rising in A Line*, "Int. J. Multiphase Flow" 2000, No. 26, pp. 1141–1181.

[5] T.S. Bellows, *The Descriptive Properties of Some Model for Density Dependence*, "Journal of Animal Ecology" 1981, Vol. 50, No. 139, pp. 139–156.

[6] M. Guzowska, *Bubble Bifurcation in the Economic Models*, „Prace Katedry Ekonometrii i Statystyki" Nr 15, Wydawnictwo Naukowe Uniwersytetu Szczecińskiego, Szczecin 2004, s. 101–112; J. Hozer, *Mikroekonometria*, PWE, Warszawa 1993.

The model

At any period of time the population (P_t) consists of majority (N_t) and minority (M_t):

$$P_t = N_t + M_t \tag{1}$$

The majority grows by natural increase and by naturalisation of members of the minority ("integration"). With a given birth and death rates of the majority population ($b^N \geq 0$, $d^N \geq 0$) with the rate of integration ($c_t \geq 0$), ratio of members of minority who change their status and become a member of the majority – the size of the majority in the next period is determined by:

$$N_{t+1} = \left(1 + b^N - d^N\right) \cdot N_t + c_t \cdot M_t \tag{2}$$

where $1 + b^N - d^N > 0$.

The minority grows by natural increase and immigration and declines by naturalisation of former members. With a given birth and death rates of the minority populations ($b^M \geq 0$, $d^M \geq 0$) and with the rate of immigration of members of the minority (m_t), the size of the minority in the next period is given by:

$$M_{t+1} = \left(1 + b^M - d^M + m_t - c_t\right) \cdot M_t \tag{3}$$

where $1 + b^M - d^M + m_t - c_t > 0$.

The ratio between minority and majority is:

$$x_t = \frac{M_t}{N_t} \tag{4}$$

Equations (2) – (4) lead to the following difference equation:

$$x_{t+1} = \frac{1 + b^M - d^M + m_t - c_t}{1 + b^N - d^N + c_t \cdot x_t} \cdot x_t \tag{5}$$

Let us assume a Cobb-Douglas production function with three inputs. The aggregate output (Y_t) is produced by physical capital (K_t), skilled labour and unskilled labour. Skilled labour is supplied by the majority and unskilled labour by the minority. In addition, we assume a constant stock of capital ($K_t = K$) and constant return of scale with respect to the skilled and unskilled labour.

$$Y_t = K \cdot M_t^{\alpha} \cdot N_t^{1-\alpha}, \; 0 < \alpha < 1 \tag{6}$$

The wages for skilled labour [$(w_N)_t$] and unskilled labour [$(w_M)_t$] are determined by their marginal products, while the wage in the sending country [$(w_s)_t$] is constant.

$$(w_M)_t = \frac{\alpha \cdot K \cdot N_t^{1-\alpha}}{M_t^{1-\alpha}} = \frac{\alpha \cdot K}{x_t^{1-\alpha}}$$

$$(w_N)_t = \frac{(1-\alpha) \cdot K \cdot M_t^{\alpha}}{N_t^{\alpha}} = (1-\alpha) \cdot K \cdot x_t^{\alpha} \tag{7}$$

$$(w_S)_t = (w_S) = cons\tan t$$

The rate of migration is an increasing function of the ratio between the wage of unskilled labour in the receiving country and the wage in the sending country. For simplification we neglect any costs of migration and specify the migration function by:

$$m_t = \tilde{m} \cdot \frac{\left[\frac{(w_M)_t}{(w_S)}\right]^{\varepsilon} - 1}{\left[\frac{(w_M)_t}{(w_S)}\right]^{\varepsilon} + 1} \tag{8}$$

The parameter ($\tilde{m}$ > 0) defines the maximum rate of migration and the parameter ($\varepsilon > 0$) determines the shape of $f\left((w_M)/(w_S)\right)$ at $(w_M)/(w_S) = 1$.

Notice that our model includes the possibility of re-emigration. Migrants return to their home country when the wage of unskilled labour in he receiving country falls below the minority wage in the sending country.

Our hypotheses are expressed by the function:

$$c_t = \tilde{c} \cdot \underbrace{\frac{\left[\frac{(w_M)_t}{(w_S)}\right]^\eta - 1}{\left[\frac{(w_M)_t}{(w_S)}\right]^\eta + 1}}_{C_1} \cdot \underbrace{e^{-\frac{\left[\frac{(w_M)_t}{(w_N)_t} - 1\right]^2}{2\cdot\sigma}}}_{C_2} \tag{9}$$

The factor C_1 shows the relationship between the migrants endeavour for integration and his relative wage $(w_M)_t/(w_S)$. The factor C_2 reflects the effects of the migrant's qualification for integration and his utility from integration: $C_2 = 1$ for $(w_M)_t = (w_N)_t \cdot C_2$ decreases with higher absolute value of the wage differential.

Equations (5), (7), (8), and (9) define the discrete dynamic system of this section.

$$x_{t+1} = \frac{1 + b^M - d^M + \tilde{m} \cdot \frac{\left[\frac{\alpha \cdot K}{x_t^{1-\alpha} \cdot (w_S)}\right]^\varepsilon - 1}{\left[\frac{\alpha \cdot K}{x_t^{1-\alpha} \cdot (w_S)}\right]^\varepsilon + 1} - \tilde{c} \cdot \frac{\left[\frac{\alpha \cdot K}{x_t^{1-\alpha} \cdot (w_S)}\right]^\eta - 1}{\left[\frac{\alpha \cdot K}{x_t^{1-\alpha} \cdot (w_S)}\right]^\eta + 1} \cdot e^{-\frac{\left[\frac{\alpha}{(1-\alpha)\cdot x_t} - 1\right]^2}{2\cdot\sigma}}}{1 + b^N - d^N + \tilde{c} \cdot \frac{\left[\frac{\alpha \cdot K}{x_t^{1-\alpha} \cdot (w_S)}\right]^\eta - 1}{\left[\frac{\alpha \cdot K}{x_t^{1-\alpha} \cdot (w_S)}\right]^\eta + 1} \cdot e^{-\frac{\left[\frac{\alpha}{(1-\alpha)\cdot x_t} - 1\right]^2}{2\cdot\sigma}} \cdot x_t} \cdot x_t \tag{10}$$

Analysis of dynamics

As an example we will consider the case when parameters take the following values:

$\alpha = 0.1 \quad K = 1 \quad w_S = 0.5 \quad \tilde{m} = 0.5$

$\tilde{c} = 0 \quad \varepsilon = 5 \quad b^N - d^N = 0 \quad b^M - d^M = 0.005.$

It means that the phenomenon of change of status by the minority does not occur, and additionally the population growth of the majority equals to 0.

These assumptions cause that equation (10) can be written as follows:

$$x_{t+1} = \left(1 + b^M - d^M + \tilde{m} \cdot \frac{\left[\dfrac{\alpha \cdot K}{x_t^{1-\alpha} \cdot (w_S)} \right]^{\varepsilon} - 1}{\left[\dfrac{\alpha \cdot K}{x_t^{1-\alpha} \cdot (w_S)} \right]^{\varepsilon} + 1} \right) \cdot x_t \qquad (11)$$

The ratio x is influenced by the birth and death rates of the minority ($b^M - d^M$) and the migration m_t.

The main part of the iteration function (11) can be locally increasing or decreasing with x. A higher minority ratio x reduces both the relative wage of the minority compared to the wage of the majority and the relative wage of the minority compared to the wage in the country of origin. A lower wage of the minority makes the receiving country less attractive for immigrants.

We can state that the following conditions are satisfied:

1. If $b^M - d^M + \tilde{m} \le b^N - d^N$, then a positive equilibrium does not exist.
2. If $b^M - d^M + \tilde{m} > b^N - d^N > b^M - d^M - \tilde{m}$ then a positive equilibrium exists.

Because in the case of analysed parameter values the second condition is satisfied, so the system has a positive equilibrium.

Analysis of dynamics was performed for two parameters: ε and $M = b^M - d^M$.

In the first step, the influence of the parameter ε on behaviour of the model was analysed (at the above-accepted values of remaining coefficients).

The properties of the difference equation are studied by simulation techniques[7]. We simulated several possible dynamic scenarios using MathCad to demonstrate bifurcation and chaotic behaviour.

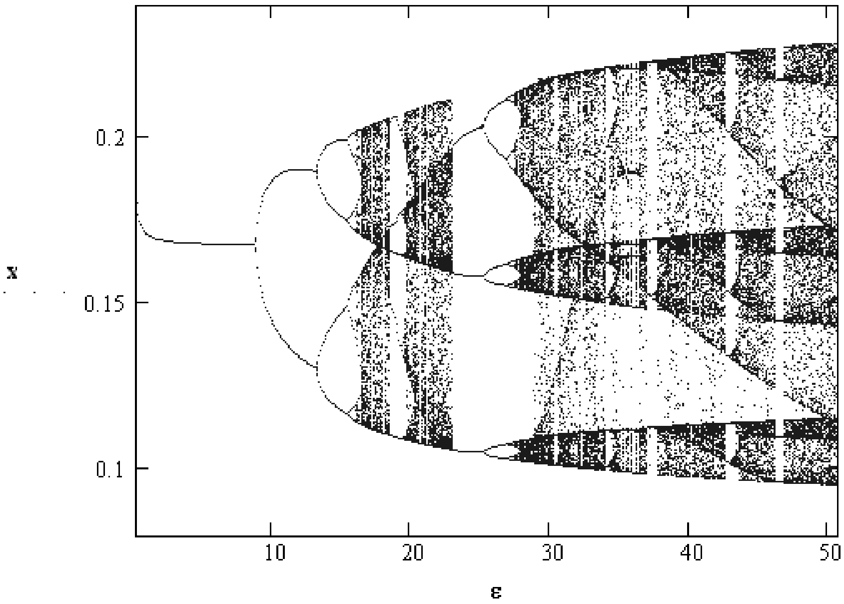
Figure 1. shows the bifurcation diagram for the parameter ε. As ε increases, the iteration function is bent around the equilibrium. For a small ε we have a unique positive stable equilibrium. As ε increases, the period doubling bifurcation emerges. If ε increases beyond a certain threshold (about 13.4), an

[7] S.N. Elaydi, *An Introduction to Difference Equations*, Springer, New York 1996; S.N. Elaydi, *Discrete Chaos*. Chapman & Hall/CRC 2000; M. Guzowska, *Non-Linear Difference Equations as a Tool of Describing and Analyzing of Chaos in Growth Theory*, International Conference Tools for Mathematical Modeling, Saint Petersburg 2001.

orbit of period three arises. For a certain range the system has the orbit of period three, we may apply the Li/Yorke theorem:[8] The dynamic system exhibit *transient chaos*.

With a view on the bifurcation diagram we conjecture, that there is chaos for ε around 13.4. It was confirmed by a comparison of the bifurcation diagram (Figure 1) with the graph of the Lyapunov exponent for the discussed system (Figure 2).

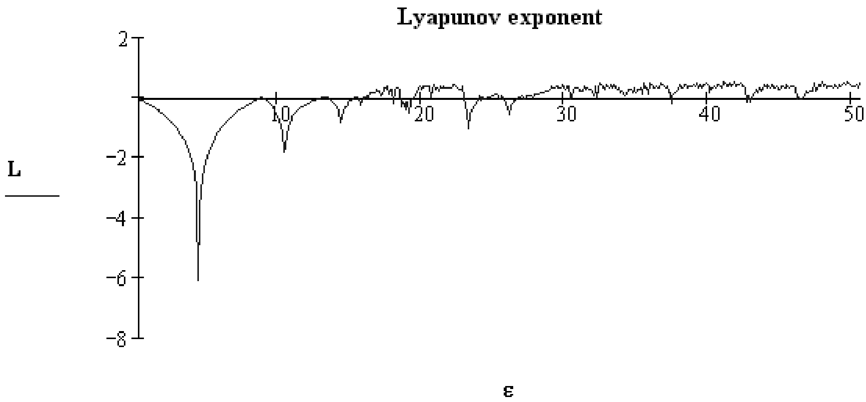
Figure 1. Bifurcation diagram for function, with varying ε parameter



Source: the author's calculations.

[8] S.N. Elaydi, *Discrete Chaos*, *op.cit.*

Figure 2. Lyapunov exponent for function, with varying ε parameter

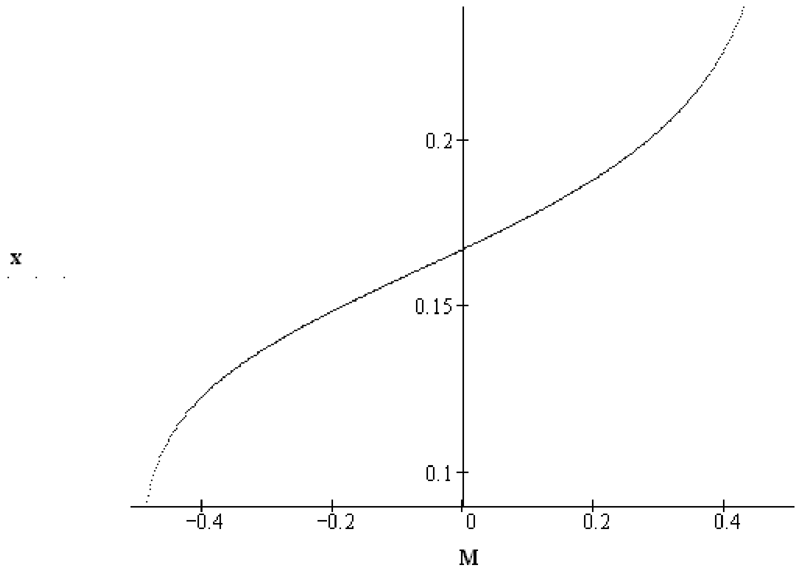


Source: the author's calculations.

Knowing behaviour of the system with relation to the parameter ε, we can analyse behaviour of the model with relation to the parameter M (simultaneously considering results obtained for ε). Therefore we obtain:

1. For $\varepsilon = 8$ (the case when with regard to the parameter ε the ratio between minority and majority is stable and convergent to the level of equilibrium), we still receive stable behaviour for $M \in (-0.5;\ 0.5)$.

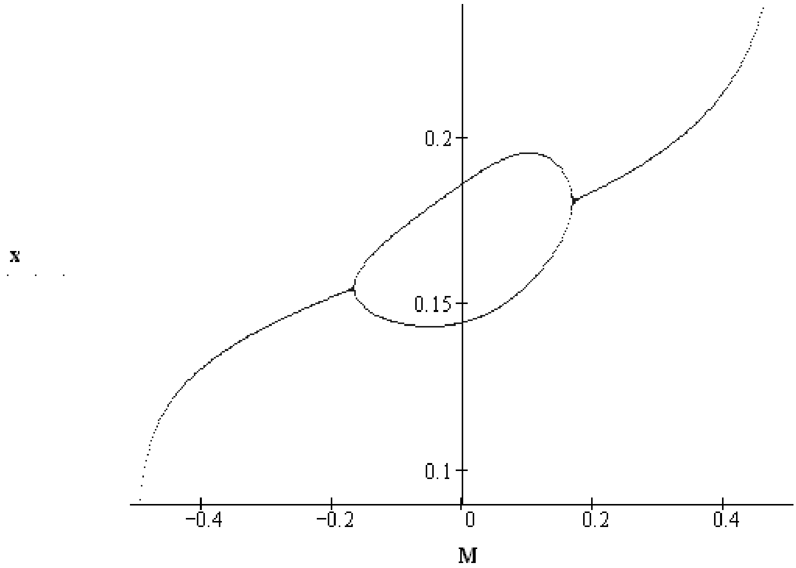
Figure 3. Bifurcation diagram for function, with varying *M* parameter, and for assumed value $\varepsilon = 8$



Source: the author's calculations.

2. For $\varepsilon = 10$ (the case when with regard to the parameter ε first bifurcations appear, that is the case when the model converges to two levels of equilibrium), we receive – with regard to the parameter M – the convergence to level of equilibrium for $M \in (-0.5;\ -0.25) \cup (0.25.;\ 0.5)$, however for $M \in (-0.25;\ 0.25)$ the divergences begin to appear (convergence to two levels of equilibrium).

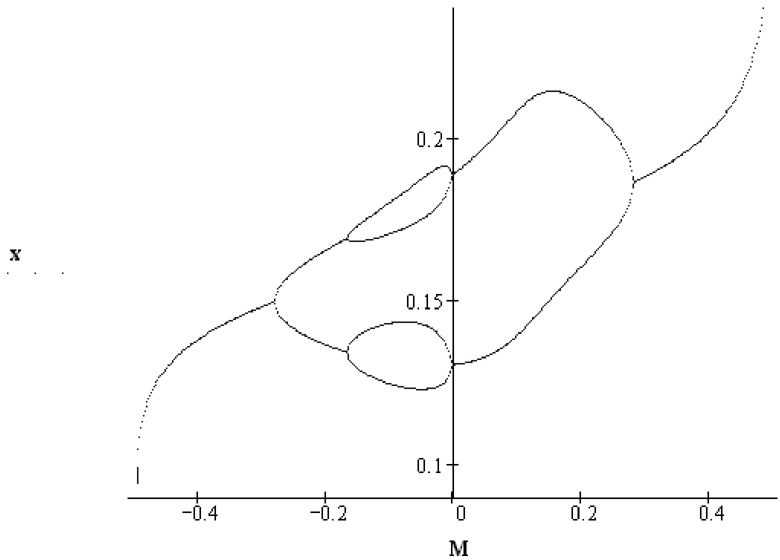
Figure 4. Bifurcation diagram for production function, with varying *M* parameter, and for assumed value $\varepsilon = 10$



Source: the author's calculations.

3. For $\varepsilon = 13$ (the case when with regard to the parameter ε second bifurcations appear, that is the case when the model converges to two levels of equilibrium), we receive – with regard to the parameter M – the convergence to level of equilibrium for $M \in (-0.5;\, -0.28) \cup (0.28.;\, 0.5)$, however for $M \in (-0.25;\, 0.25)$ the divergences begin to appear (convergence to forth levels of equilibrium).

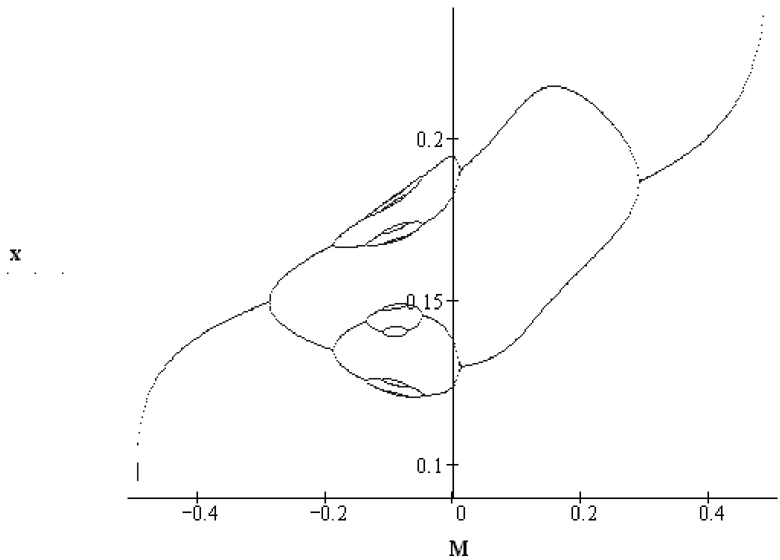
Figure 5. Bifurcation diagram for production function, with varying *M* parameter, and for assumed value $\varepsilon = 13$



Source: the author's calculations.

4. For $\varepsilon = 13.4$ (the case when with regard to the parameter ε the model exhibit beginning of chaotic behaviour), we receive – with regard to the parameter the M – the convergence to the level of equilibrium for $M \in (-0.5;\, -0.29) \cup (0.29.;\, 0.5)$, but for $M \in (-0.29;\, -0.29)$ similarly as for received parameter ε, the chaotic behaviour arises.

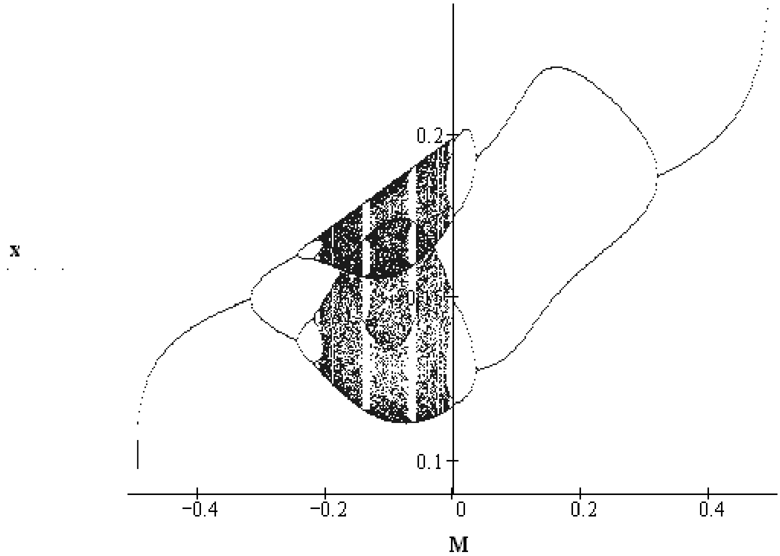
Figure 6. Bifurcation diagram for production function, with varying *M* parameter, and for assumed value $\varepsilon = 13.4$



Source: the author's calculations.

5. For $\varepsilon = 15$ (the case when with regard to the parameter ε the model exhibit chaotic behaviour), we receive – with regard to the parameter the M – the convergence to the level of equilibrium for $M \in (-0.5;\, -0.31) \cup (0.31.;\, 0.5)$, but for $M \in (-0.31;\, -0.31)$ similarly as for received parameter ε, the chaotic behaviour arises.

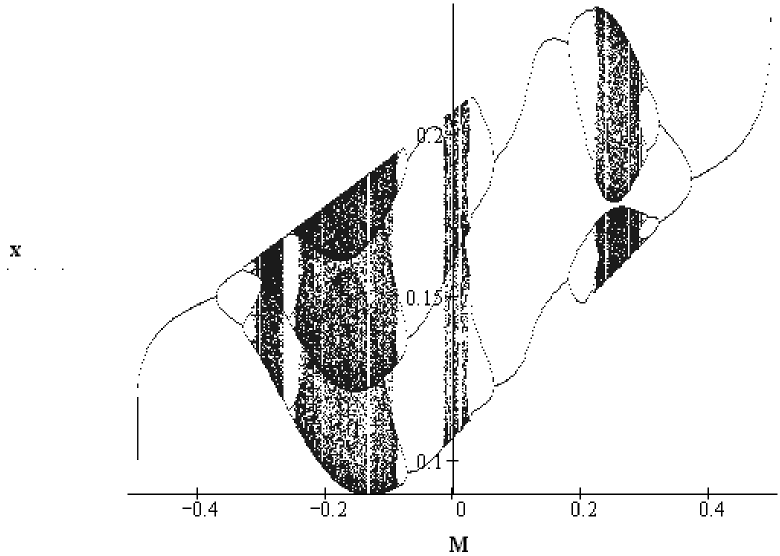
Figure 7. Bifurcation diagram for production function, with varying *M* parameter, and for assumed value $\varepsilon = 15$



Source: the author's calculations.

6. For $\varepsilon = 20$ (the case when with regard to the parameter ε the model exhibit chaotic behaviour), we receive – with regard to the parameter the M – the convergence to the level of equilibrium for $M \in (-0.5;\ -0.36) \cup (0.38.;\ 0.5)$, but for $M \in (-0.36;\ -0.38)$ similarly as for received parameter ε, the chaotic behaviour arises.

Figure 8. Bifurcation diagram for production function, with varying *M* parameter, and for assumed value $\varepsilon = 15$



Source: the author's calculations.

Observing the results of the simulation it is easy to notice that the dynamic system is structurally unstable. Changes of the parameter values can lead to bifurcation. This implies that minor variations of the parameter values can completely alter the dynamics. Small variations of the birth or death rates, of the wage in the sending country or of other parameters can destroy existing equilibrium or cause the emergence of a new equilibrium.

Conclusions

Discussed model explains the rates of immigration and integration (1) by the relative wages of the minority compared with the wage of the majority (2) and by the relative wage of the minority compared with the wage in the country of origin. The relative wages of the minority are again negatively related to the relative size of the minority. These links involve both stabilising and destabilising forces. A higher *x,* for instance, reduces the relative wage of the minority. The lower wage discourages immigration and this in turn is stabilising. But at the same time the lower relative wage of the minority makes assimilation more difficult and thereby destabilises the dynamics.

The dynamic model is structurally unstable. Therefore a minor change of one single parameter can produce a completely different (qualitative) shape of the path of x. Furthermore, we give an example for the incidence of chaos, i.e. the time path of x is very sensitive to the initial conditions.

Due to the non-linear nature of the dynamics it is impossible to forecast the time path of the relative size of the minority (x). The simultaneous operation of stabilising and destabilising forces does not allow us to predict the forces predominating in the long run.

Additionally, appearance of the bubble bifurcation can be interpreted in the discussed case as the self-controlling system. In spite of existing chaotic behaviour (for the parameter $\varepsilon > 13.5$), at the increase of the population growth ratio for the minority (as well as with its decrease – negative values), chaotic behaviour of the system stabilises.

Such kind of behaviour of the minority to majority ratio can be facilitation during trials of controlling the migration processes.

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BUBBLING I BISTABILNOŚĆ W MODELU IMIGRACJI I INTEGRACJI

Streszczenie

Artykuł jest próbą analizy mechanizmu powstawania chaotycznych zachowań bistabilnych oraz bifurkacji typu „bubble" dla grupy modeli opisanych nieliniowymi równaniami różnicowymi, z dwoma parametrami decyzyjnymi. Jest też próbą odpowiedzi na pytanie, które z tych zachowań jest bardziej typowe dla układów ekonomicznych.

Powyższe analizy zostały przeprowadzone na przykładzie modelu imigracji i integracji.

Tłumaczenie Małgorzata Guzowska

Słowa kluczowe: bistabilność, bifurkacja, model imigracji i integracji.