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DISCRETENESS OF TIME AND CHANGE

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1. INTRODUCTION. TIME AND CHANGE – TWO CORRELATED NOTIONS

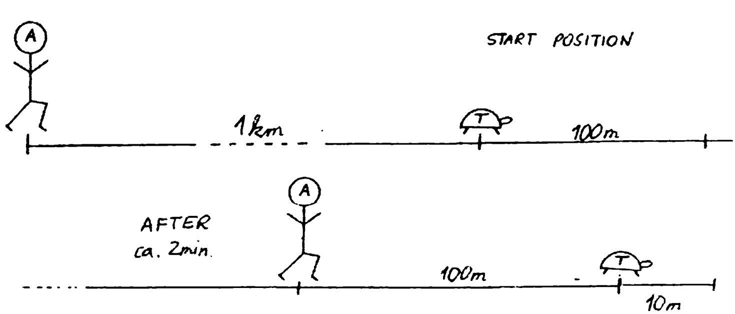
Time and change are two notions, where each seems to presuppose the other. To measure the flow of time one has to consider changes (like the movements of the hand of a clock), on the other hand changes suppose time as a frame in which they can happen. To ask which notion is the more basic one is a 'chicken-or-egg' question. However, there seems to be a tendency to consider time as more fundamental, taking it like space as a basic category. There may be many good reasons for this. In everyday life one can have the subjective impression that time is going on and nothing happens. In the structure of indoeuropean languages, time is already built into the grammar of verbs, which usually are used to describe changes, movements and events. (The role of time in the grammar of natural languages may have been partly responsible for the emergence of temporal logics, perhaps also in connection with Montague's idea of a universal grammar. But it took a very long time until a logic of the dual notion of *change* has been developed.) In science, like classical physics, time and space constitute the frame in which changes (movements) are described. Here it is not even necessarily supposed that there are changes at all.[1] Moreover, in ancient times, arguments were already formulated against the existence of changes resp. movements. These paradoxes were usually directly connected to a special understanding of infinity. Many of their modern solutions have made use of the notion of actual infinity. However we concentrate on a solution, which is not based on so strong assumptions about this kind of infinity.

2. ARGUMENTS FOR THE DISCRETENESS OF TIME

2.1. PARADOX OF ACHILLES AND THE TORTOISE; ITS STANDARD SOLUTION

Maybe the most famous of these arguments is that of Achilles and the tortoise, which is attributed to Zenon of Elea. Since Achilles was known to be very fast when running it seems to be a trivial outcome that he will win in a race against a tortoise, known to be very slow, even if he starts some distance behind it. For, if Achilles reaches the point where the tortoise started, it is ahead again at some point, and when Achilles reaches this point, the tortoise will be ahead again, and so on. So he can never catch it.

Fig. 1



[1] The situation is of course different in the case of the theory of relativity.

To describe the standard solution, offered by classical mathematics, let us assume for simplicity that Achilles is precisely 10-times faster than the tortoise and that it starts 1km ahead. When Achilles reaches its starting point, it will be 1/10 km ahead, and when he comes to this point, it will be 1/100 km ahead, and so on. The standard solution is that the infinite series

$$\sum_{n=0}^{\infty} \frac{1}{10^n} = 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots$$

converges and its limit is 10/9 which is 1,1111... km.

To see this, lets assume that

$$s = 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots$$

(where on the right side of = stands an infinitely long term).
Let us multiply this equation by 10 to get the equation

$$10s = 10 + 1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \ldots = 10 + s$$

(here we used the arithmetical law of distributivity).
The equation 10s = 10 + s leads to 9s = 10
and has therefore as solution s = 10/9.

But what did we assume here? We were calculating with infinite long terms and did apply arithmetical laws like distributivity and associativity, and we assumed that this infinite long term *has* already a value s (so we proceeded impredicatively). To see that such a method can lead to odd results, let us consider the infinite sum

t = 1 + (-1) + 1 + (-1) + 1 + ...

usually written as

t = 1 – 1 + 1 – 1 + 1 – 1 + ...

By associativity we get

t = (1 – 1) + (1 – 1) + (1 – 1) + ... = 0 + 0 + 0 + ... = 0

and also

t = 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + ... = 1 + 0 + 0 + 0 + ... = 1 which means 0 = 1, a nice contradiction. (It may be noticed that G. Grandy in 1710 simply took t = ½. Even Euler wrote in 1740 things like 1 – 3 + 5 – 7 + 9 – … = 0.)[2]

This shows that even our rather elementary calculation of s needs some further justification. Moreover, it is totally unrealistic to assume that Achilles is precisely k-times faster than the tortoise (where k is some constant number greater than 1); in general Achilles and the tortoise will even not move uniformly. In any case one has to employ some deeper mathematical theorems like that of Bolzano-Weierstraß, that every monotone and bounded sequence of real numbers has a limit. Here is the basic assumption that the real numbers form a continuum (and are suitable for modeling the continuous flow of time). Furthermore, this concept of real numbers was developed only in the 19th century and is based on a strong notion of actual infinity which itself rests on the fundamental ideas that infinite sets exist and that infinite processes can be carried out „until the end". The assumptions and methods of the formerly used calculus of infinitesimals introduced by Newton and Leibniz are even far more problematic and questionable and were already sharply and correctly criticized by George Berkeley, that such an odd argumentation wouldn't be allowed in theology: "May we not call them (the infinitesimals) the ghosts of departed quantities …?"[3]

2.2. PARADOXES OF ACTUAL INFINITY

To show that the assumption of the existence of actual infinity is problematic, let us consider the following paradox (which is folklore amongst mathematicians and may have been inspired originally by that of Achilles and the tortoise):

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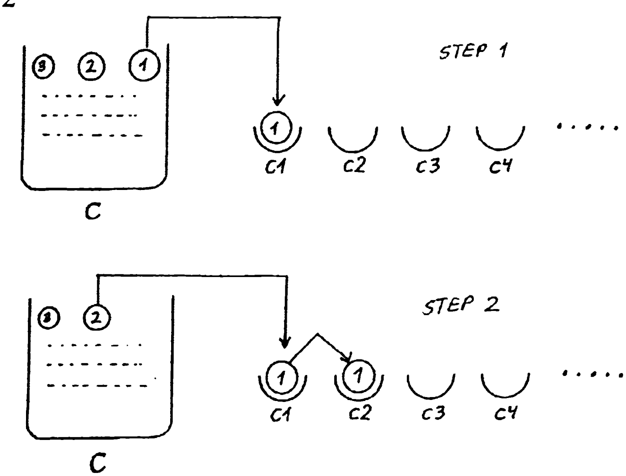
[2] G. Grandi, *De infinitis infinitorum infinitique parvorum ordinibus* (1710); L. Euler, *De seriebus quibusdam considerationes* (1740); both found in: H. Meschkowski, *Problemgeschichte der Mathematik II*, BI-Wissenschaftsverlag Mannheim/Wien/Zürich 1981, 82, 127.

[3] G. Berkeley, *The Analyst or a Discourse addressed to an Infidel Mathematician*, London 1734.

Let us assume that there is a big can **C** in which there are all natural numbers. Let us further assume that there are infinitely many small cans **c1**, **c2**, **c3**, ... each of which gives room for exactly one natural number.

Fig. 2



Now let us start the following procedure:

In the first step, we put number 1 out of **C** and give it **c1**. In the second step, we put number 1 out of **c1** to give it in **c2** and we take number 2 out of **C** to give it in **c1**. As a consequence of step *n* there will be the number n in **c1**, number n-1 in **c2**, number n-2 in **c3**, ..., and number 1 in **cn**. In the step *n+1* we put every number which is in a small can in the next one and put number n+1 in **c1**. Now imagine that we have carried out this infinite process. The question now is: Where are all these natural numbers?

They've vanished! They cannot be in the big can, because they were taken out of it. But all the small cans also are empty. Assume that in a small can there is a number k. It has been put there in a step, but taken out of it in the next and never put in again. (Of course, you can speak about sets instead of cans if you don't believe that there are so many, and such big cans **C**, **c1**, **c2**, **c3**, ...)

As in the case of the paradox of Achilles and the tortoise, this paradox arises if an infinite process is actually completed. There are many paradoxes connected with actual infinity, one of the most famous ones being that of Banach-Tarski.[4] This one rests, like many important mathematical theorems such as that of Bolzano-Weierstraß, on the Axiom of Choice (AC) – a principle, which is trivially true in case of finite sets but the object of discussion if transferred to infinite sets.[5] These paradoxes may be used as arguments against the platonistic position concerning the existence of mathematical objects, like the classical continuum of real numbers, and in consequence also against the continuity of time.

But how does one solve else the Achilles-tortoise-paradox? If one assumes discrete time with sufficiently small units, after a finite number of „moments" there will be one in which Achilles and the tortoise both will have reached the same point. If one would make a movie with 24 pictures per second, there will be a last picture on which the tortoise will be ahead, on the next one Achilles will be at least at the same point as the tortoise, possibly already further. Supposing discrete time sets

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[4] See e.g. S. Wagon, *The Banach-Tarski-Paradox*, Cambridge University Press 1986.

[5] It is well known that this axiom cannot be refuted (Gödel 1936) or proved (Cohen 1963) on the basis of the usual set-theoretical axioms (e.g. these of Zermelo-Fraenkel, ZF). So set theory, and in consequence mathematics in general, may be split into two versions (like geometry in a Euklidean and Non-Euklidean one). To get some arguments for the one or the other one, one can consult the book of H. Herrlich, *Axiom of Choice*, Springerverlag, Amsterdam 2006, with its chapters *Disasters with AC* and *Disasters without AC*. One alternative axiom to AC is the Axiom of Determinateness (AD), introduced by J. Mycielski and H. Steinhaus (*A mathematical axiom contradicting the Axiom of Choice*, Bull. Acad. Polon. Sci. Ser. Sci. Mat. Astr. Phys. 10(1962), 1-3), which is also transferring a situation, which is trivially true in the finite, to infinity. With the help of AD one can prove that every set is measurable and that therefore the Banach-Tarski paradox cannot be constructed. "If a model of ZF satisfies AD, then this model is closer to physical reality than any model of ZF+AC. For example, the Banach-Tarski paradoxical decomposition of a ball is impossible." (W. Marek, J. Mycielski, *Foundations of mathematics in the twentieth century.* Amer. Math. Monthly 108(2001), 449-468). Again we encounter the question of which mathematical models are closer to some reality.

us free from the strong philosophical and mathematical assumptions of actual infinity.

3. A FORMAL DESCRIPTION OF A DISCRETE STRUCTURE OF TIME BY LOGIC OF CHANGE

It seems to be obvious that it cannot be decided whether „real time“ is continuous or discrete. In our culture, with the background of our languages and natural sciences, it may be that it is more natural to consider time as continuous. At least this looks to be a very useful, elegant and economical assumption for describing changes which we also may consider as continuous, even if we take that simply as a fiction or idealization. After all, it seems that a supposition that time is discrete does not stay in any contradiction with our experience – we do not observe any continuous (or even dense) structures independently of the fact that we construct still more and more precise tools for divisions and analysis of physical objects. It would be perhaps easier to accept this point of view if we would speak about time in terms of changing events. This way of treating time is already known in philosophy from Aristotle and in particular from Leibniz. Moreover we also know about such a practice in natural languages – e.g. in Hopis' language. We accept the Sapir-Whorf-thesis, if only in that it may be more natural for the Hopi-Indians to consider changes to be prior to time which is thought to be discrete, because, according to Whorf, in their language time is often expressed by referring to events.[6]

As we are going to show, the specified intuitions of speaking about (discrete) time in terms of changing events – actually: generalizations of these intuitions – may be properly described by a modal system LC.[7] However LC as a formal characterization of discrete structure of

[6] Cf. B. L. Whorf, *Language, Thought and Reality*, ed. by John B. Carroll, The M.I.T. Press, Cambridge 1956. For a critical discussion, cf. E. Malotki, *Hopi Time: A Linguistic Analysis of Temporal Concepts in the Hopi Language*, Mouton, Berlin 1983.

[7] Which is originally described by K. Świętorzecka in *Classical Conceptions of the Changeability of Situations and Things Represented in Formalized Languages*, ed. by CSWU, Warsaw 2008 and further elaborated by K. Świętorzecka, J. Czermak in *Some*

time does not yield the ontological decision about priority of (some sort of) change to time. This is clear in view of the fact that logic LC may also be axiomatizied by axioms with primitive temporal operator "and next" or with two primitive operators "and next" and "it changes that...".

3.1. TIME DEFINED BY CHANGE – SYSTEM LC

The proposed analysis is based on classical sentential logic. As we have already stated we consider the idea that *the flow* of time – so actually, time by itself – may be described just by changes of some events, or we could say in other words that: *time is passing just when sequences of changing events occur in reality*. To make precise this way of speaking let us establish that a change consists in the appearance or disappearance of situations. We take the symbol A to represent a sentence which refers to (describes) some situation which may appear or disappear. We would say that:

(*) *time is passing* iff *there is some A changing its truth value*

and

(**) *A changes its truth value* means that:
at first it is A and next it is not A or at first it is not A and next it is A

Taking any discrete set of truth values – and we ground our analysis on the classical logic which is characterized by matrices over {Truth, False} – we interpret "first" and "next" in frame of a discrete structure. So, for some moment n we may draw:

			n		n+1
			A		¬A
(***)	*A changes*	iff:		or	
			¬A		A

Calculus for a Logic of Change, Journal of Applied Non-Classical Logic (2011) (accepted for printing).

Equivalences (*), (**) and (***) lead us to the following explication:

(PS) *time is passing* iff *there is some A changing its truth value,* that is:
there is some moment n:
in n it is A and in n+1 it is ¬ A
or
in n it is ¬ A and in n+1 it is A

This way of speaking about time as dependent on change is actually proposed in the mentioned system LC.

The calculus LC is expressed in the sentential language built up from propositional constants out of $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$, classical connectives: ¬, ∧, ∨, →, ↔ and one-place modal operator C to be read as *it changes that......*

Formulas are defined inductively as follows:

$$A ::= \alpha_i \mid \neg A \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid A \leftrightarrow A \mid CA \qquad \text{for every } i \in N$$

To describe the stated logic of change we assume as axioms all classical sentential tautologies and all formulas of the following forms:

A1. $CA \rightarrow C\neg A$
(*If* A *is changing, then not-*A *is also changing.*)

A2. $C(A \wedge B) \rightarrow CA \vee CB$
(*If the complex of situations:* A *and* B *is changing, then at least one of its components has to change.*)

A3. $\neg A \wedge B \wedge CA \wedge \neg CB \rightarrow C(A \wedge B)$
(*If it is not the case that* A *but this changes and it is the case that* B *and this is not changing, then the subject of change is:* A *and* B.)

A4. $\neg A \wedge \neg B \wedge CA \wedge CB \rightarrow C(A \wedge B)$
(*If it is not the case that* A *but this changes, and it is not the case that* B *and this also changes, then the subject of change is:* A *and* B.)

and we also take basic rules:
(*Modus ponens*) $A, A \rightarrow B \vdash B$,
($\neg$C-rule) $A \vdash \neg CA$ (*Theorems don't change*),
(Replacement) $A[B], B \leftrightarrow B' \vdash A[B']$

Lets now interpret the above-described calculus LC semantically by so-called *histories of changes*. We consider a function φ which assigns to every natural number n a subset of the set of all elementary propositions. In temporal interpretation of the function φ, the value of φ(n) may be understood as the set of these elementary sentences, which are true in some moment n of time.

To speak about validity of LC formulas in terms of such functions we use the relation $\vDash$.

The expression $\varphi \vDash^n A$ we read: A *is true in moment* n *in some history* φ. We define inductively:

Definition ($\vDash$) For any φ and $n \in N$

(i) $\varphi \vDash^n \alpha_i$ iff $\alpha_i \in \varphi(n)$

Let A, B be formulas, then:

(ii) $\varphi \vDash^n \neg A$ iff $\varphi \nvDash^n A$,
(iii) $\varphi \vDash^n A \wedge B$ iff $\varphi \vDash^n A$ and $\varphi \vDash^n B$,
(iv) $\varphi \vDash^n A \vee B$ iff $\varphi \vDash^n A$ or $\varphi \vDash^n B$,
(v) $\varphi \vDash^n A \rightarrow B$ iff $\varphi \nvDash^n A$ or $\varphi \vDash^n B$,
(vi) $\varphi \vDash^n A \leftrightarrow B$ iff ($\varphi \nvDash^n A$ or $\varphi \vDash^n B$) and ($\varphi \vDash^n A$ or $\varphi \nvDash^n B$),
(vii) $\varphi \vDash^n CA$ iff ($\varphi \vDash^n A$ and $\varphi \nvDash^{n+1} A$) or ($\varphi \nvDash^n A$ and $\varphi \vDash^{n+1} A$).

It is important to note that LC is complete in this semantics, which means that all and only formulas true in every history φ on every moment n are LC-theses (completeness theorem). In frame of our formalism we may say that the passing of time explicated in (PS) consists in changing the truth value of some formula A in sense of operator C (cf. (vii)).

However the flow of time may be also described with the use of temporal operator N read as "next it is ...." and understood as follows:

(viii) $\varphi \vDash^n NA$ iff $\varphi \vDash^{n+1} A$.

The equivalence:

(N/C) $NA \leftrightarrow (\neg A \leftrightarrow CA)$

(*Next it is* A *iff (it is not the case that* A *iff* A *changes its value)*)
which may be considered as a definition of N.

As it is already shown the same logic may be characterized by axioms expressed in the sentential language built as it was in case of LC with operator N as a primitive symbol instead of operator C.

This time we take as axioms all propositional classical tautologies and formulas of the shape:

A1*. $N\neg A \leftrightarrow \neg NA$
(*Next it is not A iff it is not the case that next it is A*)

A2. $N(A\rightarrow B) \rightarrow (NA\rightarrow NB)$,
(If next it is the case that A implies B, then if next it is A then next it is B)

and also take also as basic rules:
(*Modus ponens*) $A, A\rightarrow B \vdash B$,
(N-rule) $A \vdash NA$ (*Theorems will also be theorems next*),
(Replacement) $A[B], B\leftrightarrow B' \vdash A[B']$

and we add as the definition of C:

$$\text{(C/N)}\quad CA \leftrightarrow (A \leftrightarrow \neg NA)$$
(A *changes iff (it is* A *iff next it is not the case that* A*)*)

By means of classical logic, we may notice that the (C/N) is equivalent of

$$\text{(C/PS')}\quad CA \leftrightarrow (\neg A \wedge NA) \vee (A\wedge\neg NA)$$

and this brings us again back to (PS) – our explication of (discrete) passing of time based on the notion of change.

The described calculus LN is a known temporal logic, already developed by A. Prior in 1956, who named it F. However it is interesting that LN is definitionally equivalent to LC and this equivalence actually shows the dual character of change and time as expressed by C and N.

Moreover, the connection is in this sense symmetric that in LN changes are *measured* by time in the same way as time is *measured* by changes in LC – the definition of N operator (N/C) is equivalent by means of classical logic, to the following formula corresponding to (C/PS'):

$$\text{(N/PS')} \quad NA \leftrightarrow (\neg A \wedge CA) \vee (A \wedge \neg CA)$$

Let us emphasize that nowhere in our approach did we assume actual infinity.[8] We only used the potential infinity of natural numbers in their discrete order. Our proposal also provides the opportunity to solve the paradox of Zenon in a much simpler way than it is by the above-described, so-called standard solution.

Consider again the competition between Achilles and tortoise in this context.

Let α_7 be: *Achilles is behind the tortoise.*

Then we will have a history φ and some natural number k with

$\varphi \models^n \alpha_7$ for all natural numbers n with $1 \leq n \leq k$

and

$\varphi \models^k \neg\alpha_7$

which means

$\varphi \models^{k-1} C\alpha_7$ and $\varphi \models^{k-1} N\neg\alpha_7$

Of course there is no paradoxical situation at all.[9]

This text is suported by Polish Ministry of Science and Higher Education (NN10116240).

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[8] That actual infinity is not necessarily inherent in mathematics (as potential infinity seems to be) can be seen by the fact that this assumption is explicitly stated as a special axiom of set theory.

[9] It is obvious that despite the admissibility of ($\neg$C-rule) and (N-rule) the corresponding implications: $A \rightarrow \neg CA$ and $A \rightarrow NA$ are not valid in both LC and LN. So from α_7 it can neither be inferred that $\neg C\alpha_7$ nor $N\alpha_7$.

NIECIĄGŁOŚĆ CZASU I ZMIANY

Streszczenie

Standardowe rozwiązanie paradoksów Zenona – w szczególności paradoksu Achillesa i żółwia – opiera się na pewnych matematycznych założeniach dotyczących liczb rzeczywistych. Wymaga ono między innymi tego, by przyjąć założenie, zgodnie z którym czas jest przynajmniej gęstym zbiorem momentów czasowych. Można jednak zaproponować także inne rozwiązanie, które opiera się na założeniu o istnieniu „atomów" czasu - w tym przypadku czas byłby uważany za strukturę dyskretną. Tego rodzaju punkt widzenia wydaje się naturalny, gdy przyjmiemy, iż czas powinien być charakteryzowany przez ciągi zdarzeń. Wówczas upływ czasu można byłoby definiować przez odwoływanie się do zachodzenia zmian. Idea pierwszeństwa zmienności względem czasu jest obecna w filozofii już od starożytności. Jednym ze współczesnych narzędzi opisu takiej koncepcji może być zdaniowa logika zmiany LC, która nie wikła Achillesa i żółwia w paradoks Zenona.

Słowa kluczowe: dyskretność czasu, zmiana, logika zmiany